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The Reversible Measures for Symmetric Nearest-Particle Systems

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Symmetric nearest-particle systems are certain spin systems on $\{0, 1\}^{\mathbb{Z}}$ in which the flip rate is a function of the distances to the nearest particle of different type to the left and right. The process differs from the ordinary nearest-particle system in that the rates are preserved if zeros and ones are interchanged. The only reversible measure for the symmetric nearest-particle system is a "renewal-type" measure (the natural analog to the nonsymmetric case). Also as in the nonsymmetric case, reversibility only occurs when the rates are of a specific form. By imposing additional conditions on the rates it can be shown that the reversible measure is the only translation-invariant, invariant measure which concentrates on configurations having infinitely many zeros and ones to either side of the origin. This can be used to prove that for a large class of translation-invariant initial distributions, weak limits are reversible measures. Then we can conclude that the process is convergent for several examples of initial distributions.

KEY WORDS: Nearest-particle systems; symmetric particle system; reversible measures; symmetric renewal measure.

1. INTRODUCTION

In 1977 Frank Spitzer introduced the nearest-particle system, which is an infinite-range generalization of the one-dimensional contact process. The contact process had been introduced by Ted Harris in 1974 and became the subject of exhaustive study. Both are examples of one-dimensional spin systems, continuous-time Markov processes η_t on $\{0, 1\}^{\mathbb{Z}}$ whose evolution is described by the rates $c(x, \eta)$, $x \in \mathbb{Z}$, with which $\eta(x)$ flips to $1 - \eta(x)$. The contact process and this nearest-particle system have rates which

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are not symmetric in zeros and ones; the rate with which a site flips is dependent on the local configuration of ones. When $\eta(x)=0$ the rates for the nearest-particle system are a function of the distance to the nearest 1 to the left and right of x. When $\eta(x)=1$ the (death) rates for the typical nearest-particle system are as they are for the contact process, identically one.

As with the contact process, one of the main questions we are interested in answering about the nonsymmetric nearest-particle system is when does the process survive and when does it die out. This involves analyzing the invariant measures. Studying a subclass of the invariant measures, the reversible measures, proves to be quite helpful. An invariant measure for a Markov process is reversible if the stationary process obtained by using it as the initial distribution is symmetric in time. As it turns out, reversible measures for the nearest-particle system are renewal measures, and they exist only when the rates are of a special form.

The essential difference between the symmetric and nonsymmetric nearest-particle systems is that in the nonsymmetric case the transition rates only depend on the distances to the nearest ones, while in the symmetric version they depend on the distances to the nearest discrepancy. Let $\{\beta(l, r), 1 \leqslant l, r \leqslant \infty\}$ be a collection of nonnegative numbers satisfying

$$\sup_{l,r} \beta(l, r) < \infty, \qquad \beta(l, r) = \beta(r, l)$$

$$\beta(1, \infty) = \beta(\infty, 1) > 0, \qquad \beta(\infty, \infty) = 0$$

For $x \in \mathbb{Z}$ and $\eta \in X = \{0, 1\}^{\mathbb{Z}}$, we define $l_x(\eta)$ and $r_x(\eta)$ by

$$l_x(\eta) = x - \sup\{y < x : \eta(y) \neq \eta(x)\}$$

and

$$r_x(\eta) = \inf\{y > x : \eta(y) \neq \eta(x)\} - x$$

with usual conventions regarding $\sup\{\varnothing\}$, $\inf\{\varnothing\}$, and arithmetic involving ∞. The rates for the symmetric nearest-particle system are then given as

$$c(x, \eta) = \beta(l_x(\eta), r_x(\eta))$$

The symmetric nearest-particle system is a generalization of a finite-range process which has been studied by Ted Cox and Rick Durrett, the threshold voter model. The threshold voter model, being a symmetric version of the contact process, has two natural trivial invariant measures, the point masses on the identically one and identically zero configurations. Most of the research done on the threshold voter model concerns the question of whether coexistence or clustering occurs. We say that there is coexistence if a nontrivial invariant measure exists; otherwise, we say that there is clustering. Thus inspired, we will try to characterize as much as possible the invariant measures for the symmetric nearest-particle system, and try to determine when coexistence and clustering occurs.

In constructing the infinite symmetric nearest-particle system the 01 (or 10) configuration plays much the same role that 1 does for the ordinary nearest-particle system. In the ordinary nearest-particle system the flip rate at a site x only depends on the configuration up to the first 1 to the left and right of the site x. To construct the symmetric system it is necessary to restrict the process to the subset

$$\tilde{X}=\left\{\eta\in X\colon \sum_{x\geqslant 0}\eta(x)=\sum_{x\leqslant 0}\eta(x)=\infty \right.$$
$$\left.\text{and } \sum_{x\geqslant 0}[1-\eta(x)]=\sum_{x\leqslant 0}[1-\eta(x)]=\infty\right\}$$

of the space X, for basically the same reason we need to restrict to the subset consisting of infinitely many ones to either side of the origin in the ordinary case. The construction then follows the model for the nonsymmetric nearest-particle system almost exactly; see Chapter VII, Section 3 in Liggett.[9] In this case, however, the finite approximations are based on fixing a 01 configuration to the left and right of the finite set instead of fixing ones, since a particle does not need to know the values to the other side of a 01 in determining the flip rate. Making the obvious modifications using 01 in place of 1, the construction becomes straightforward if one also notes that the death rate of a 01 is bounded since the rates are bounded.

As with the ordinary system, we can begin the analysis of the invariant measures for the symmetric nearest-particle system by characterizing the subclass of reversible measures. If $S(t)$ is the semigroup for the symmetric nearest-particle system concentrating on $\tilde{X}$, we say that a probability measure μ on $\tilde{X}$ is reversible for the process if

$$\int fS(t)\,g\,d\mu=\int gS(t)\,f\,d\mu \qquad \forall f, g\in C(\tilde{X})$$

where $C(\tilde{X})$ is the collection of all bounded, continuous functions on $\tilde{X}$. As it turns out, reversible measures for the symmetric nearest-particle system have a form we will call a symmetric renewal measure (see μ_β in Theorem 1.1 below), and they exist when the rates are of a special form.

The following theorem, which we prove in Section 2, is the analog of a result by Spitzer on the ordinary nearest-particle system.

Theorem 1.1. There exists a reversible measure for the symmetric nearest-particle system concentrating on $\tilde{X}$ if and only if $\beta(l, r)$ is of the form

$$\beta(l, r) = \frac{F(l+r)}{F(l)\,F(r)} \qquad \forall l \geqslant 1, \qquad r > 1 \tag{1.2}$$

for some positive function F, and setting

$$g(k) = \left(\frac{F(2)}{\beta(1, 1)}\right)^{1/2} \frac{1}{F(k+1)}, \qquad k \geqslant 1$$

$\exists \theta > 0$ such that $\sum_{k=1}^{\infty} g(k)\,\theta^k = 1$ and $\sum_{k=1}^{\infty} kg(k)\,\theta^k < \infty$.

If this is the case, the reversible measure is given by

$$\mu_\beta(\overbrace{10\cdots 0}^{k_1}\overbrace{11\cdots 1}^{l_1}0\cdots 0\overbrace{1\cdots 1}^{l_m}\overbrace{0\cdots 0}^{k_{m+1}}1)$$
$$= \frac{\beta(k_1)\,\beta(k_2)\cdots\beta(k_{m+1})\,\beta(l_1)\,\beta(l_2)\cdots\beta(l_m)}{2\alpha} \tag{1.3}$$

for $k_i, l_i \geqslant 1$, and $m \geqslant 0$, where $\beta(k) = g(k)\,\theta^k$ is a probability density on the positive integers with finite mean α.

Remark. Note that if F is positive and the power series $\sum_k [1/F(k+1)]\,x^k$ has a nonzero radius of convergence R, then for $\beta(l, r)$ of the form (1.2), Theorem 1.1 says there is a constant $0 < c \leqslant \infty$ such that there exists a reversible measure for $\beta(1, 1) < c$, but not for $\beta(1, 1) > c$. Furthermore, $c < \infty$ if and only if $\sum_k [R^k/F(k+1)] < \infty$.

In Section 3 we impose some conditions on the rates $\beta(l, r)$ and use the free energy technique to show that considering only measures which concentrate on $\tilde{X}$ the reversible measures are just the translation-invariant, invariant measures. This analog of Liggett's result for the ordinary system relies on our characterization of the reversible measures in the previous theorem. We will require $\beta(l, r)$ to satisfy (1.2), with F positive, and that $\beta(l, r)$ be monotone decreasing in l and r for large $l+r$. This monotonicity assumption is really quite natural, since it is a generalization of the condition for attractive rates.

Theorem 1.4. Assume that $\beta(l, r)$ satisfies (1.2), where F is positive, and that there exists a positive integer N such that $\beta(l, r)$ is monotone decreasing in l and r for $l+r \geqslant N$. Then any invariant, translation-invariant probability measure on $\tilde{X}$ is reversible and, hence by Theorem 1.1, a symmetric renewal measure.

In order to obtain some interesting applications of Theorem 1.4 it will be useful to realize a few facts about weak limits of Cesaro averages and invariant measures. At the end of Section 3 we show that in the attractive case the process can be extended continuously to include the configurations $\eta \equiv 0$ and $\eta \equiv 1$. Suppose μ is a probability measure on X which concentrates on a set of configurations for which the semigroup $S(t)$ can be defined continuously. Assume that a weak limit of Cesaro averages of $\mu S(t)$ is a measure μ^* which also concentrates on a set of configurations for which the semigroup can be defined continuously. Denoting the convergent sequence as μ_n, by Skorohod's theorem there exist random variables η_n and η and a probability space Ω so that η_n has distribution μ_n for each n, η has distribution μ^*, and $\eta_n \to \eta$ a.s. on Ω. Thus $S(t)\, f(\eta_n) \to S(t)\, f(\eta)$ as $n \to \infty$ for all bounded continuous functions f. Hence,

$$ES(t)\, f(\eta_n) \to ES(t)\, f(\eta)$$

and

$$\int S(t)\, f\, d\mu_n \to \int S(t)\, f\, d\mu^*$$

This result is enough to give us the invariance of μ^* if the proof of Proposition 1.8 in Chapter I of Liggett[9] is followed. Suppose now that ν is any invariant measure on X such that $\nu(\tilde{X}) > 0$. Let ν_c be the measure ν conditioned to concentrate on $\tilde{X}$. Then for any cylinder set A and $t > 0$,

$$\begin{aligned} \int 1_A\, d\nu_c &= \frac{\int 1_{A \cap \tilde{X}}\, d\nu}{\nu(\tilde{X})} = \frac{\int 1_{A \cap \tilde{X}}\, d\nu_t}{\nu_t(\tilde{X})} \\ &= \frac{\int_{\tilde{X}} S(t)\, 1_A\, d\nu}{\nu(\tilde{X})} = \int S(t)\, 1_A\, d\nu_c \end{aligned}$$

where the third equality follows because $\eta_0 \in \tilde{X}$ implies $\eta_t \in \tilde{X}$. Hence ν_c is invariant. To get a consequence of this, assume the hypotheses of Theorem 1.4 and let $\nu_{1/2}$ denote the product measure with density 1/2. Then if a weak limit ν^* of Cesaro averages of $\nu_{1/2} S(t)$ concentrates on $\tilde{X}$, it must be a symmetric renewal measure, while in the attractive case if no reversible measure exists, ν^* must be the measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Further applications are found below.

If we have rates $\beta(l, r)$ of the same form as in Theorem 1.4, then it can be shown that for a large class of translation-invariant initial distributions any weak limit which concentrates on $\tilde{X}$ is reversible. Thus if the process is reversible (a reversible measure exists), a large class of initial distributions converge to the symmetric renewal measure determined by the rates, while if the process is not reversible, we get many instances for which initial distributions which are symmetric in zeros and ones converge to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. The proof of these facts will be found in Section 4.

It is conjectured that the remark following Theorem 1.1 generalizes beyond the reversible case: Under certain conditions on the rates, for fixed $\beta(l, r)$, $l + r \geqslant 3$, we can find a constant $0 < c \leqslant \infty$ so that there exists a nontrivial invariant measure for $\beta(1, 1) < c$. Under other conditions on the rates, for fixed $\beta(l, r)$, $l + r \geqslant 3$, we can find a constant $c < \infty$ so that only trivial invariant measures exist if $\beta(1, 1) > c$. These questions of coexistence and clustering have been partially solved at this time. The complete solution is still an open problem.

2. CHARACTERIZATION OF THE REVERSIBLE MEASURES

In this section we use two propositions to prove Theorem 1.1.

Proposition 2.1. Suppose that $\beta(l, r)$ satisfies (1.2), where F is some positive function. Set

$$g(k) = \left(\frac{F(2)}{\beta(1, 1)}\right)^{1/2} \frac{1}{F(k+1)}$$

and suppose $\exists \theta > 0$ so that $\beta(k) = g(k)\,\theta^k$ is a probability density function on $\{1, 2, \ldots\}$ with finite mean α. Then μ_β as given by (1.3) is reversible for the symmetric nearest-particle system concentrating on $\tilde{X}$.

Proof. We begin by considering more carefully the approximating Markov chains which were used in the construction of the process. For $m \leqslant n$, let $Z_{m,n} = \{m, m+1, \ldots, n\}$ and $X_{m,n} = \{0, 1\}^{Z_{m,n}}$. Let $\eta_t^{m,n}$ be the approximating process defined on $X_{m,n}$ conditioned on fixing a 0, 1 at the sites $m-2$, $m-1$ and $n+1$, $n+2$. Set $\mu_{m,n}$ to be the probability measure on $X_{m,n}$ which is defined by

$$\begin{aligned}\mu_{m,n}\{\zeta\} = \mu_\beta\{\eta\colon \eta = \zeta \text{ on } Z_{m,n} \mid \eta(m-2) = \eta(n+1) = 0 \\ \text{and } \eta(m-1) = \eta(n+2) = 1\}\end{aligned}$$

Our first goal is to prove reversibility of $\mu_{m,n}$ for the approximating process. For $x \in Z_{m,n}$ and $\zeta \in X_{m,n}$, we must show

$$\frac{\mu_{m,n}(\zeta)}{\mu_{m,n}(\zeta_x)} = \frac{\beta(l_x(\zeta_x), r_x(\zeta_x))}{\beta(l_x(\zeta), r_x(\zeta))} \tag{2.2}$$

where $\zeta_x(y) = \zeta(y)$ for $y \neq x$ and $\zeta_x(x) = 1 - \zeta(x)$.

Suppose that $\zeta(x) = 1$.

Case A. ζ contains a configuration of the form

$$1\overbrace{0\cdots0}^{k_1}\underset{\text{site } x}{1}\overbrace{0\cdots0}^{k_2}1, \qquad k_i \geqslant 1$$

Then

$$\frac{\mu_{m,n}(\zeta)}{\mu_{m,n}(\zeta_x)} = \frac{\beta(k_1)\,\beta(1)\,\beta(k_2)}{\beta(k_1+k_2+1)} = \frac{g(k_1)\,g(1)\,g(k_2)}{g(k_1+k_2+1)}$$

$$= \frac{\dfrac{F(2)}{\beta(1,1)}\,\dfrac{1}{F(k_1+1)}\,\dfrac{1}{F(k_2+1)}\,\dfrac{1}{[\beta(1,1)\,F(2)]^{1/2}}}{\left(\dfrac{F(2)}{\beta(1,1)}\right)^{1/2}\dfrac{1}{F(k_1+k_2+2)}}$$

$$= \frac{\dfrac{F(k_1+k_2+2)}{F(k_1+1)\,F(k_2+1)}}{\beta(1,1)} = \frac{\beta(k_1+1,k_2+1)}{\beta(1,1)}$$

Case B. ζ contains a configuration of the form

$$1\overbrace{0\cdots0}^{k}\underset{x}{1}\overbrace{1\cdots1}^{l}0, \qquad k \geqslant 1, \qquad l \geqslant 2$$

Then

$$\frac{\mu_{m,n}(\zeta)}{\mu_{m,n}(\zeta_x)} = \frac{\beta(k)\,\beta(l)}{\beta(k+1)\,\beta(l-1)} = \frac{g(k)\,g(l)}{g(k+1)\,g(l-1)}$$

$$= \frac{\dfrac{1}{F(k+1)\,F(l+1)}}{\dfrac{1}{F(k+2)\,F(l)}} = \frac{\dfrac{F(k+2)}{F(k+1)\,F(1)}}{\dfrac{F(l+1)}{F(l)\,F(1)}} = \frac{\beta(k+1,1)}{\beta(1,l)}$$

Case C. ζ contains a configuration of the form

$$0\overbrace{1\cdots1}^{l_1}\underset{\text{site } x}{1}\overbrace{1\cdots1}^{l_2}0, \qquad l_i \geqslant 1$$

Then

$$\frac{\mu_{m,n}(\zeta)}{\mu_{m,n}(\zeta_x)} = \frac{\beta(l_1+l_2+1)}{\beta(l_1)\,\beta(1)\,\beta(l_2)} = \frac{g(l_1+l_2+1)}{g(l_1)\,g(1)\,g(l_2)}$$

$$= \frac{\left(\dfrac{F(2)}{\beta(1,1)}\right)^{1/2} \dfrac{1}{F(l_1+l_2+2)}}{\dfrac{F(2)}{\beta(1,1)}\,\dfrac{1}{F(l_1+1)}\,\dfrac{1}{F(l_2+1)}\,\dfrac{1}{[F(2)\,\beta(1,1)]^{1/2}}}$$

$$= \frac{\beta(1,1)}{\dfrac{F(l_1+l_2+2)}{F(l_1+1)\,F(l_2+1)}} = \frac{\beta(1,1)}{\beta(l_1+1,l_2+1)}$$

Hence (2.2) is satisfied. By symmetry, (2.2) also holds if we assume $\zeta(x)=0$. Let $S_{m,n}(t)$ denote the semigroup corresponding to the process $\eta_t^{m,n}$, and $C_{m,n}$ be the set of all functions on $X_{m,n}$. Since, by (2.2), $\mu_{m,n}$ is reversible for $\eta_t^{m,n}$, we have

$$\int f S_{m,n}(t)\, g\, d\mu_{m,n} = \int g S_{m,n}(t)\, f\, d\mu_{m,n} \qquad \forall f, g \in C_{m,n} \tag{2.3}$$

We need to check that

$$\lim_{\substack{m\to-\infty\\ n\to+\infty}} \mu_{m,n} = \mu_\beta$$

in the topology of weak convergence on X. By the renewal property of μ_β [which means given $\eta(x)=0$ (1) and $\eta(x+1)=1$ (0), the random variables $\{\eta(k), k\leqslant x\}$ are conditionally independent of the random variables $\{\eta(k), k>x\}$], for the limit on n it suffices to show that

$$\mu_\beta(01\overbrace{\cdots|}^{n\text{ sites}}01) \to \frac{1}{2\alpha} \qquad \text{as} \quad n\to\infty$$

Now

$$\mu_\beta(01\overbrace{\cdots|}^{n}01) = \frac{\mu_\beta(01\overbrace{\cdots}^{n}01)}{\mu_\beta(01)} = \frac{\mu_\beta(01\overbrace{\cdots}^{n}1) - \mu_\beta(01\overbrace{\cdots}^{n}11)}{\mu_\beta(01)}$$

Similarly

$$\mu_\beta(01\overbrace{\cdots\mid 1}^{n}0)=\frac{\mu_\beta(01\overbrace{\cdots 1}^{n}0)}{\mu_\beta(10)}=\frac{\mu_\beta(01\overbrace{\cdots 1}^{n})-\mu_\beta(01\overbrace{\cdots 1}^{n}1)}{\mu_\beta(10)}$$

Thus

$$\mu_\beta(01\overbrace{\cdots\mid 0}^{n}1)-\mu_\beta(01\overbrace{\cdots\mid 1}^{n}0)=\mu_\beta(\overbrace{1\cdots\mid 1}^{n}0)-\mu_\beta(\overbrace{1\cdots\mid}^{n-1}10)\to\tfrac{1}{2}-\tfrac{1}{2}=0$$

by left\right symmetry and the Alternating Renewal Theorem. Let μ be the ordinary renewal measure determined by the density β. Then for $k_i, l_i\geqslant 1$ and $m\geqslant 0$

$$\begin{aligned}&\mu_\beta(1\overbrace{0\cdots 0}^{k_1}\overbrace{11\cdots 1}^{l_1}0\cdots 0\overbrace{1\cdots 1}^{l_m}\overbrace{0\cdots 0}^{k_{m+1}}1)\\&=\tfrac{1}{2}\mu(1\overbrace{0\cdots 0}^{k_1-1}1\overbrace{0\cdots 0}^{l_1-1}1\cdots 1\overbrace{0\cdots 0}^{l_m-1}1\overbrace{0\cdots 0}^{k_{m+1}-1}1)\end{aligned}$$

So applying the regular Renewal Theorem, we get

$$\begin{aligned}\frac{1}{\alpha}&=\lim_n \mu(1\overbrace{\cdots\mid\cdots}^{n}1)\\&=\lim_n\frac{2\mu_\beta(01\overbrace{\cdots 0}^{n}1)+2\mu_\beta(01\overbrace{\cdots 1}^{n}0)}{2\mu_\beta(01)}\\&=\lim_n 2\mu_\beta(01\overbrace{\cdots\mid 0}^{n}1)\end{aligned}$$

which gives the desired result. The limit as $m\to-\infty$ is obtained in basically the same way. Now, by the construction, $S_{m,n}(t)\,g\to S(t)\,g$ as $m\to-\infty$ and $n\to\infty$ uniformly on compact subsets of $\tilde{X}$ for each $g\in C(\tilde{X})$, and since $\mu_\beta(\tilde{X})=1$, we can take the limit in (2.3) to get

$$\int fS(t)\,g\,d\mu_\beta=\int gS(t)\,f\,d\mu_\beta$$

for all f, $g\in\mathfrak{D}$, where $\mathfrak{D}$ is the set of functions on X which depend on only finitely many coordinates. Therefore, since $\mathfrak{D}$ is dense in $C(\tilde{X})$, we can conclude that μ_β is reversible. ∎

The next statement is basically the converse of Proposition 2.1.

Proposition 2.4. Assume that there exists a reversible measure μ on $\tilde{X}$ for the symmetric nearest-particle system with strictly positive rates $\beta(l, r)$. Then there exists a positive function F such that

$$\beta(l, r) = \frac{F(l+r)}{F(l)\,F(r)} \qquad \text{for} \quad l \geqslant 1, \quad r > 1$$

Furthermore, setting

$$g(k) = \left(\frac{F(2)}{\beta(1, 1)}\right)^{1/2} \frac{1}{F(k+1)} \qquad \text{for} \quad k \geqslant 1$$

we have that $\exists \theta > 0$ so that $\beta(k) = g(k)\,\theta^k$ is a probability density on the positive integers with finite mean, and $\mu = \mu_\beta$.

Proof. The first step is to find equations which the function $\beta(\cdot, \cdot)$ satisfies. By making a slight modification in the proof of Proposition 2.7 of Chapter IV in Liggett,[9] we get

$$\mu\{\eta(x) = 1 \mid \eta(y),\, y \neq x\} = \frac{\beta(l_x(\eta), r_x(\eta))}{\beta(l_x(\eta), r_x(\eta)) + \beta(l_x(\eta_x), r_x(\eta_x))} \tag{2.5}$$

Since $\beta(\cdot, \cdot)$ is strictly positive, (2.5) implies that μ assigns positive probability to any subset of X which depends on only finitely many coordinates. For $x \in \mathbb{Z}$ and $k_1,..., k_n \geqslant 1$, define subsets of X by

$$\begin{aligned} A_x(k_1,..., k_n) &= \{\eta\colon \eta(x) = \eta(x+k_1) = \cdots = \eta(x+k_1+\cdots+k_n) = 1; \\ &\quad \text{and } \eta(y) = 0 \text{ for all other } x < y < x+k_1+\cdots+k_n, \text{ and } y = x-1\} \end{aligned}$$

By (2.5)

$$\frac{\mu(A_x(j, k))}{\mu(A_x(j+k))} = \frac{\beta(j, k)}{\beta(1, 1)}$$

for all $x \in \mathbb{Z}$ and all $j, k > 1$. Using (2.5) again, write

$$\begin{aligned} \frac{\mu(A_x(j, k, l))}{\mu(A_x(j+k+l))} &= \frac{\mu(A_x(j, k, l))}{\mu(A_x(j+k, l))} \frac{\mu(A_x(j+k, l))}{\mu(A_x(j+k+l))} \\ &= \frac{\beta(j, k)\,\beta(j+k, l)}{\beta(1, 1)\,\beta(1, 1)} \qquad \text{for} \quad j, k, l > 1 \end{aligned}$$

A similar argument gives that this quantity also equals

$$\frac{\beta(k, l)\beta(j, k+l)}{\beta(1, 1)\,\beta(1, 1)}$$

Now

$$\begin{aligned}\frac{\mu(A_x(j, k, 1))}{\mu(A_x(j+k+1))} &= \frac{\mu(A_x(j, k, l))}{\mu(A_x(j+k))}\,\frac{\mu(A_x(j+k, 1))}{\mu(A_x(j+k+1))}\\ &= \frac{\beta(j, k)\,\beta(j+k, 1)}{\beta(1, 1,)\,\beta(1, 2)} \qquad \text{for} \quad j, k > 1\end{aligned}$$

A similar argument gives that this quantity also equals

$$\frac{\beta(k, 1)\,\beta(j, k+1)}{\beta(1, 2)\,\beta(1, 1)}$$

Thus we have for $k \neq 1$

$$\beta(j, k)\,\beta(j+k, l) = \beta(k, l)\,\beta(j, k+l) \tag{2.6}$$

Now for $l > 1$

$$\frac{\mu(A_x(1, 1, l))}{\mu(A_x(l+2))} = \frac{\mu(A_x(1, 1, l))}{\mu(A_x(2, l))}\,\frac{\mu(A_x(2, l))}{\mu(A_x(l+2))} = \frac{\beta(1, 1)\,\beta(2, l)}{\beta(2, 2)\,\beta(1, 1)}$$

Also

$$\frac{\mu(A_x(1, 1, l))}{\mu(A_x(l+2))} = \frac{\mu(A_x(1, 1, l))}{\mu(A_x(1, l+1))}\,\frac{\mu(A_x(1, l+1))}{\mu(A_x(l+2))} = \frac{\beta(1, l)\,\beta(1, l+1)}{\beta(3, 1)\,\beta(2, 1)}$$

Thus we have

$$\beta(3, 1)\,\beta(2, 1)\,\beta(2, l) = \beta(2, 2)\,\beta(1, l)\,\beta(1, l+1)$$

or

$$\frac{\beta(1, l)\,\beta(1, l+1)}{\beta(2, l)} = \frac{\beta(3, 1)\,\beta(2, 1)}{\beta(2, 2)} = c \qquad \forall l \geqslant 2 \tag{2.7}$$

where c is a constant. Now

$$\beta(2, 2) = \frac{\beta(3, 1)\,\beta(2, 1)}{c}$$

and

$$\beta(3, 2) = \frac{\beta(4, 1)\,\beta(3, 1)\,\beta(2, 1)}{c\beta(2, 1)}$$

Assume $i, j \geqslant 2$, and

$$\beta(i, j) = \frac{\beta(j+i-1, 1)\,\beta(j+i-2, 1)\cdots\beta(j, 1)}{c\beta(i-1, 1)\,\beta(i-2, 1)\cdots\beta(2, 1)} \tag{2.8}$$

Then

$$\beta(i, j+1) = \frac{\beta(i+j, 1)\,\beta(i, j)}{\beta(j, 1)} = \frac{\beta(j+i, 1)\,\beta(j+i-1, 1)\cdots\beta(j+1, 1)}{c\beta(i-1, 1)\,\beta(i-2, 1)\cdots\beta(2, 1)}$$

Hence, by induction, (2.8) holds for all $i, j \geqslant 2$. Let $F(i) = \beta(i-1, 1)\cdots\beta(2, 1)c$, for $i > 2$, $F(2) = c$, and $F(1) = 1$. Then we have

$$\beta(l, r) = \frac{F(l+r)}{F(l)\,F(r)} \qquad \forall l \geqslant 1, \quad r > 1 \tag{2.9}$$

Next we want to characterize the reversible measure μ. For k_i, $l_i \geqslant 1$ and $m \geqslant 0$, using (2.5), write

$$\frac{\mu(\underset{x_0}{1}\overbrace{0\cdots0}^{k_1}\overbrace{\underset{x_1}{1}\,\underset{x_2}{1}\cdots1}^{l_1}0\cdots0\overbrace{1\cdots\underset{x_{n-2}}{1}\;\;\underset{x_{n-1}}{1}}^{l_m}\overbrace{0\cdots0}^{k_{m+1}}\underset{x_n}{1})}{\mu(\underset{x_0}{1}\overbrace{0\cdots0}^{k_1}\overbrace{\underset{x_1}{1}\,\underset{x_2}{1}\cdots1}^{l_1}0\cdots0\overbrace{1\cdots\underset{x_{n-2}}{1}}^{l_m-1}\overbrace{0\cdots0}^{k_{m+1}+1}\underset{x_n}{1})} = \frac{\beta(x_{n-1}-x_{n-2}, x_n-x_{n-1})}{\beta(l_m, 1)}$$

and

$$\frac{\mu(\underset{x_0}{1}\overbrace{0\cdots0}^{k_1}\overbrace{\underset{x_1}{1}\,\underset{x_2}{1}\cdots1}^{l_1}0\cdots0\overbrace{1\cdots\underset{x_{n-2}}{1}}^{l_m-1}\;\overbrace{0\cdots0}^{k_{m+1}+1}\;\underset{x_n}{1})}{\mu(\underset{x_0}{1}\overbrace{0\cdots0}^{k_1}\overbrace{\underset{x_1}{1}\,\underset{x_2}{1}\cdots1}^{l_1}0\cdots0\overbrace{1\cdots\underset{x_{n-3}}{1}}^{l_m-2}\;\overbrace{0\cdots0}^{k_{m+1}+2}\;\underset{x_n}{1})} = \frac{\beta(x_{n-2}-x_{n-3}, x_n-x_{n-2})}{\beta(l_m-1, 1)}$$

Continue in this manner to finally get

$$\mu(\underset{x_0}{1}\overbrace{0\cdots0}^{k_1}\overbrace{\underset{x_1}{1}\,\underset{x_2}{1}\cdots1}^{l_1}0\cdots0\overbrace{1\cdots\underset{x_{n-2}}{1}\;\;\underset{x_{n-1}}{1}}^{l_m}\overbrace{0\cdots0}^{k_{m+1}}\underset{x_n}{1})$$

$$= \frac{\mu(\underset{x_0}{1}0\cdots0\underset{x_n}{1})\,\beta(x_{n-1}-x_{n-2}, x_n-x_{n-1})\cdots\beta(x_1-x_0, x_n-x_1)}{\prod_{i=1}^{l_1}\beta(i, 1)\cdots\prod_{i=1}^{l_m}\beta(i, 1)}$$

$$= \frac{\mu(\underset{x_0}{1}0\cdots0\underset{x_n}{1})\dfrac{F(x_n-x_{n-2})}{F(x_n-x_{n-1})\,F(x_{n-1}-x_{n-2})}\cdots\dfrac{F(x_n-x_0)}{F(x_n-x_1)\,F(x_1-x_0)}}{\dfrac{\beta(1, 1)}{c}F(l_1+1)\cdots\dfrac{\beta(1, 1)}{c}F(l_m+1)}$$

$$= \frac{\mu(\underset{x_0}{1}\,0\cdots 0\,\underset{x_n}{1})\, F(x_n - x_0)}{\left(\dfrac{\beta(1,1)}{F(2)}\right)^m F(k_1+1)\cdots F(k_{m+1}+1)\, F(l_1+1)\cdots F(l_m+1)}$$

$$= \frac{\mu(\underset{x_0}{1}\,0\cdots 0\,\underset{x_n}{1})}{g(x_n - x_0 - 1)}\, g(k_1)\cdots g(k_{m+1})\, g(l_1)\cdots g(l_m)$$

where

$$g(k) = \left(\frac{F(2)}{\beta(1,1)}\right)^{1/2} \frac{1}{F(k+1)}, \qquad k \geqslant 1$$

Set

$$h_x(n) = \frac{\mu(\underset{\text{site } x}{1}\,\overbrace{0\cdots 0}^{n}\,1)}{g(n)}, \qquad n \geqslant 1$$

Then, since $\mu(\tilde{X}) = 1$,

$$h_x(n) = \sum_{k,l \geqslant 1} \frac{\mu(\underset{x}{1}\,\overbrace{0\cdots 0}^{n}\,\overbrace{1\cdots 1}^{l}\,\overbrace{0\cdots 0}^{k}\,1)}{g(n)}$$

$$= \sum_{k,l \geqslant 1} \frac{\mu(\underset{x}{1}\,\overbrace{0\cdots 0}^{n+k+l}\,1)}{g(n+k+l)}\, g(k)\, g(l) = \sum_{k,l \geqslant 1} h_x(n+k+l)\, g(k)\, g(l)$$

The remaining step in our proof involves solving for $h_x(n)$. Let H be the set of all nonnegative functions $h(n)$ on $\{1, 2, \ldots\}$, such that $h(1) = 1$ and

$$\sum_{k,l \geqslant 1} h(n+k+l)\, g(k)\, g(l) \leqslant h(n), \qquad \forall n \geqslant 1 \tag{2.10}$$

Now H is convex and compact in the topology of pointwise convergence on $[0, \infty) \times [0, \infty] \times [0, \infty) \times [0, \infty) \times \cdots$, since $h(1+k+l)\, g(k)\, g(l) \leqslant 1$ $\forall k, l$. Now H is nonempty because $h(n) = h_x(n)/h_x(1)$ is in H, for $x \in \mathbb{Z}$. Let H_0 denote the set of all h in H such that equality holds in (2.10) for all $n \geqslant 1$. Since H is metrizable, Choquet's Representation Theorem implies that every element of H has an integral representation in terms of the extreme points of H. Now the integral representation of any h in H_0 can only involve extreme points of H which are in H_0. Let h be an extreme point of H_0. Then [note that $h(n) > 0$ for all n]

$$\begin{aligned} h(n) &= \sum_{k,l \geqslant 1} h(n+k+l)\, g(k)\, g(l) \\ &= \sum_{k,l \geqslant 1} \frac{h(n+k+l)}{h(1+k+l)} h(1+k+l)\, g(k)\, g(l) \end{aligned} \tag{2.11}$$

Since

$$\sum_{k,l \geqslant 1} h(1+k+l)\, g(k)\, g(l) = h(1) = 1$$

and

$$\frac{h(\cdot + k + l)}{h(1+k+l)} \in H_0 \qquad \forall k, l$$

then (2.11) gives h as a convex combination of elements of H_0. Since h is extremal, we must have

$$\frac{h(n+k+l)}{h(1+k+l)} = h(n) \qquad \forall k, l, n \geqslant 1$$

Now

$$h(4) = h(2)\, h(3) \qquad \text{and} \qquad h(5) = [h(3)]^2$$

Also

$$h(5) = h(2)\, h(4)$$

Hence,

$$[h(3)]^2 = h(2)\, h(4) = h(2)\, h(2)\, h(3)$$

Thus

$$h(3) = [h(2)]^2 \qquad \text{and} \qquad h(4) = [h(2)]^3$$

Now suppose $h(n+k+l) = [h(2)]^{n+k+l-1}$. Then $h(n+k+l+2) = h(n+k+l)\, h(3) = [h(2)]^{n+k+l+1}$. So by induction $h(n) = [h(2)]^{n-1}$ $\forall n \geqslant 1$. Set $\theta = h(2)$. Then

$$\begin{aligned} h(1) = 1 &= \sum_{k,l \geqslant 1} h(1+k+l)\, g(k)\, g(l) \\ &= \sum_{k,l \geqslant 1} \theta^{k+l} g(k)\, g(l) = \left[\sum_{k=1}^{\infty} \theta^k g(k) \right]^2 \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} g(k)\,\theta^k = 1 \tag{2.12}$$

giving a unique $\theta = h(2)$. Hence H_0 consists of just one element, the function $h(n) = \theta^{n-1}$, θ solving (2.12). So $h_x(n) = h_x(1)\,\theta^{n-1}$, for all $x \in \mathbb{Z}$ and $n \geqslant 1$. Thus by the definition of $h_x(n)$

$$\mu(\underset{x}{1}\,\overbrace{0\cdots 0}^{n}\,1)\,g(1) = \mu(\underset{x}{1}\;0\;1)\,g(n)\,\theta^{n-1}$$

If we interchange the roles of left and right we get

$$\mu(\underset{x}{1}\,\overbrace{0\cdots 0}^{n}\,1)\,g(1) = \mu(\underset{x+n-1}{1}\;0\;1)\,g(n)\,\theta^{n-1}$$

Hence $\mu(\underset{x}{1}\;0\;1)$ is independent of x. Therefore, $h_x(n) = a\theta^n$ for some constant $a > 0$. Now we want to find $a = \mu(1\;0\;1)/g(1)\theta$. In our characterization of the measure μ so far we have not assumed anything special about the role of zeros versus that of ones. Thus it is also true that

$$\begin{aligned}
&\mu(0\,\overbrace{1\cdots 1}^{k_1}\,\overbrace{0\cdots 0}^{l_1}\,1\cdots 1\,\overbrace{0\cdots 0}^{l_m}\,\overbrace{1\cdots 1}^{k_{m+1}}\,0)\\
&\quad= \frac{\mu(0\;1\;0)}{g(1)\,\theta}\,g(k_1)\,\theta^{k_1}\cdots g(k_{m+1})\,\theta^{k_{m+1}}\,g(l_1)\,\theta^{l_1}\cdots g(l_m)\,\theta^{l_m}
\end{aligned}$$

So if we knew that $\mu(1\;0\;1) = \mu(0\;1\;0)$, then we would know that the measure μ is symmetric in zeros and ones. Now

$$\begin{aligned}
\mu(0\;1\;0) &= \sum_{k,l \geqslant 1} \mu(1\,\overbrace{0\cdots 0}^{k}\,1\,\overbrace{0\cdots 0}^{l}\,1)\\
&= \sum_{k,l \geqslant 1} \frac{\mu(1\;0\;1)}{g(1)\theta}\,g(k)\,\theta^k g(1)\,\theta g(l)\,\theta^l = \mu(1\;0\;1)
\end{aligned}$$

giving the desired symmetry. Observe that since $\mu(\underset{x}{1}\,\overbrace{0\cdots 0}^{n}\,1)$ is independent of x, we have

$$\mu(\underset{x}{1}\;1) = 1 - \sum_{n=1}^{\infty} (n+1)\,\mu(1\,\overbrace{0\cdots 0}^{n}\,1) \tag{2.13}$$

must also be independent of x. We now know

$$\tfrac{1}{2}=\mu(1)=\mu(1\,1)+\sum_{n=1}^{\infty}\mu(1\,\overbrace{0\cdots0}^{n}\,1) \tag{2.14}$$

Thus using (2.13) and (2.14), we get

$$1-\sum_{n=1}^{\infty}(n+1)\,\mu(1\,\overbrace{0\cdots0}^{n}\,1)=\tfrac{1}{2}-\sum_{n=1}^{\infty}\mu(1\,\overbrace{0\cdots0}^{n}\,1)$$

giving

$$\tfrac{1}{2}=\sum_{n=1}^{\infty}n\mu(1\,\overbrace{0\cdots0}^{n}\,1)=\sum_{n=1}^{\infty}na\theta^n g(n)$$

Hence

$$a=\frac{1}{2\sum_{n=1}^{\infty}ng(n)\,\theta^n}$$

So if we set $\beta(k)=g(k)\,\theta^k$, where θ is the unique value satisfying $\sum_{k=1}^{\infty}g(k)\,\theta^k=1$, then β is a probability density on the positive integers with finite mean α, and

$$\mu(1\,\overbrace{0\cdots0}^{k_1}\,\overbrace{1\cdots1}^{l_1}\,0\cdots0\,\overbrace{1\cdots1}^{l_m}\,\overbrace{0\cdots0}^{k_{m+1}}\,1)$$
$$=\frac{\beta(k_1)\,\beta(k_2)\cdots\beta(k_{m+1})\,\beta(l_1)\,\beta(l_2)\cdots\beta(l_m)}{2\alpha}\qquad\square$$

Consider the example in which the rates have the form

$$\beta(l,r)=\left(\frac{1}{l}+\frac{1}{r}\right)^p=\frac{(l+r)^p}{l^p r^p}$$

for $l+r\geqslant 3$, so that $F(k)=k^p$. Then by Theorem 1.1 there exists a reversible measure for the system only if

$$\beta(1,1)\leqslant 2^p\left[\sum_{k=1}^{\infty}\frac{1}{(k+1)^p}\right]^2 \tag{2.15}$$

If the inequality in (2.15) is strict, then we have a reversible masure, while if equality holds in (2.15), a reversible measure exists if and only if $p>2$.

3. THE TRANSLATION-INVARIANT, INVARIANT MEASURES

In this section we prove Theorem 1.4. We will assume that $\beta(l, r)$ satisfies

$$\beta(l, r) = \frac{F(l+r)}{F(l)\,F(r)} \qquad \forall l \geqslant 1, \quad r > 1$$

for some positive function F, and that there exists a positive integer N such that $\beta(l, r)$ is monotone decreasing in l and r for $l + r \geqslant N$. The goal is to show that any translation-invariant, invariant measure on $\tilde{X}$ must be reversible (symmetric renewal). The method that will be employed here is sometimes called the free energy technique. It has been used by Holley and Stroock in studying spin systems, and derives its name from associations with physics, where the "free energy" of a measure is one characterization of physical systems. For an example of this technique with some motivating ideas as to its use in the present setting see Liggett.[8] Now choose $\theta > 0$ so that

$$\sum_k \frac{k}{F(k+1)} \theta^k < \infty$$

and

$$\frac{F(k+1)}{\theta F(k)} \downarrow \gamma > 1 \qquad \text{as} \quad k \to \infty$$

Let

$$g(k) = \left(\frac{F(2)}{\beta(1, 1)}\right)^{1/2} \frac{1}{F(k+1)} \theta^k \qquad \text{for} \quad k \geqslant 1$$

Then

$$\frac{g(k)}{g(k+1)} \downarrow \gamma \qquad \text{as} \quad k \to \infty$$

Define ν to be the function on cylinder sets of X which is symmetric in zeros and ones, so that

$$\nu(\overbrace{0 \cdots 0}^{k_1}\ \overbrace{1 \cdots 1}^{l_1}\ 0 \cdots 0\ \overbrace{1 \cdots 1}^{l_m}\ \overbrace{0 \cdots 0}^{k_{m+1}})$$

$$= \left(\sum_{k \geqslant k_1} g(k)\right) g(k_2) \cdots g(k_m) \left(\sum_{k \geqslant k_{m+1}} g(k)\right) g(l_1)\, g(l_2) \cdots g(l_m)$$

and

$$\nu(\overbrace{0\cdots0}^{k_1}\overbrace{1\cdots1}^{l_1}0\cdots1\overbrace{0\cdots0}^{k_m}\overbrace{1\cdots1}^{l_m})$$
$$=\left(\sum_{k\geqslant k_1} g(k)\right)g(k_2)\cdots g(k_m)\,g(l_1)\,g(l_2)\cdots g(l_{m-1})\left(\sum_{l\geqslant l_m} g(l)\right)$$

for k_i, l_i, $m\geqslant 1$, with

$$\nu(\overbrace{0\cdots0}^{k})=\sum_{j\geqslant k}\ \sum_{i\geqslant j} g(i)$$

Let μ be a translation-invariant measure on $\tilde{X}$ which is symmetric in zeros and ones. In this paper we will actually prove more than is necessary for the proof of Theorem 1.4. The general statements in the following lemmas and propositions are intended for use in future results. So with that in mind we will not yet restrict μ to be invariant. Set $\mu_t=\mu S(t)$. For $n\geqslant 0$, let $Z_{0,n}=\{0, 1,..., n\}$ and $X_{0,n}=\{0, 1\}^{Z_{0,n}}$. If $x\in Z_{0,n}$ and $\eta\in X_{0,n}$, set

$$a_n^t(\eta, x)=\int_{\{\zeta:\zeta=\eta \text{ on } Z_{0,n}\}} c(x,\zeta)\,d\mu_t(\zeta)$$
$$M_n^t(\eta)=\mu_t\{\zeta:\zeta=\eta \text{ on } Z_{0,n}\}, \qquad N_n(\eta)=\nu(\eta)$$
$$B_n=\{(x,\eta)\in Z_{0,n}\times X_{0,n}:\eta(x)=1 \text{ and } \exists y, z, v \text{ such that}$$
$$y, z\in Z_{0,x-1},\ u, v\in Z_{x+1,n},$$
$$\text{and } \eta(y)=0, \eta(z)=1, \eta(u)=0, \text{ and } \eta(v)=1\}$$

Define the free energy on $[0, n]$ of the measure μ_t as

$$H_n(\mu_t)=\sum_{\eta\in X_{0,n}} M_n^t(\eta)\log\left(\frac{M_n^t(\eta)}{N_n(\eta)}\right)$$

where the function $x\log x$ is understood to be 0 at 0. Our first result enables us to write the derivative of $H_n(\mu_t)$ as a sum of terms over B_n plus a sum of negative terms.

Lemma 3.1. We have

$$\frac{dH_n(\mu_t)}{dt}=\sum_{(x,\eta)\in B_n}[a_n^t(\eta_x, x)-a_n^t(\eta, x)]\log\frac{a_n^t(\eta_x, x)\,M_n^t(\eta)\,N_n(\eta_x)}{a_n^t(\eta, x)\,M_n^t(\eta_x)\,N_n(\eta)}$$
$$-\frac{1}{2}\sum_{\substack{x\in Z_{0,n}\\ \eta\in X_{0,n}}}[a_n^t(\eta_x, x)-a_n^t(\eta, x)]\log\frac{a_n^t(\eta_x, x)}{a_n^t(\eta, x)}$$

Proof. For fixed $\eta \in X_{0,n}$, let

$$g(\zeta) = \begin{cases} 1 & \text{if } \zeta = \eta \text{ on } Z_{0,n} \\ 0 & \text{otherwise} \end{cases}$$

If we denote the generator of the process by Ω, then from the construction, $[S(s)\, g - g]/s \to \Omega g$ as $s \to 0$ uniformly on compact subsets of $\tilde{X}$, so we have

$$\frac{dM_n^t(\eta)}{dt} = \int \Omega g \, d\mu_t = \sum_{x \in Z_{0,n}} [a_n^t(\eta_x, x) - a_n^t(\eta, x)]$$

Now

$$\begin{aligned} \frac{dH_n(\mu_t)}{dt} &= \sum_{\eta \in X_{0,n}} \frac{dM_n^t(\eta)}{dt} \log \left(\frac{M_n^t(\eta)}{N_n(\eta)} \right) \\ &\quad + \sum_{\eta \in X_{0,n}} M_n^t(\eta) \frac{d}{dt} \log \left(\frac{M_n^t(\eta)}{N_n(\eta)} \right) \\ &= \sum_{\substack{x \in Z_{0,n} \\ \eta \in X_{0,n}}} [a_n^t(\eta_x, x) - a_n^t(\eta, x)] \log \left(\frac{M_n^t(\eta)}{N_n(\eta)} \right) \\ &\quad + \sum_{\substack{x \in Z_{0,n} \\ \eta \in X_{0,n}}} [a_n^t(\eta_x, x) - a_n^t(\eta, x)] \\ &= \sum_{\substack{x \in Z_{0,n} \\ \eta \in X_{0,n}}} [a_n^t(\eta_x, x) - a_n^t(\eta, x)] \log \left(\frac{M_n^t(\eta)}{N_n(\eta)} \right) \end{aligned}$$

Making the change of variable $\eta \to \eta_x$ gives for $x \in Z_{0,n}$

$$\begin{aligned} &\sum_{\eta \in X_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)] \log \frac{N_n(\eta_x)}{M_n(\eta_x)} \\ &\qquad = - \sum_{\eta \in X_{0,n}} [a_n(\eta_x, x) - a_n(\eta, x)] \log \frac{N_n(\eta)}{M_n(\eta)} \end{aligned}$$

So the previous two identities show

$$\frac{dH_n(\mu_t)}{dt} = \frac{1}{2} \sum_{\substack{x \in Z_{0,n} \\ \eta \in X_{0,n}}} [a_n(\eta_x, x) - a_n(\eta, x)] \log \frac{M_n(\eta)\, N_n(\eta_x)}{M_n(\eta_x)\, N_n(\eta)}$$

Thus,

$$\frac{dH_n(\mu_t)}{dt}=\frac{1}{2}\sum_{\substack{x\in Z_{0,n}\\ \eta\in X_{0,n}}}[a_n'(\eta_x,x)-a_n'(\eta,x)]\log\frac{a_n'(\eta_x,x)\,M_n'(\eta)\,N_n(\eta_x)}{a_n'(\eta,x)\,M_n'(\eta_x)\,N_n(\eta)}$$

$$-\frac{1}{2}\sum_{\substack{x\in Z_{0,n}\\ \eta\in X_{0,n}}}[a_n'(\eta_x,x)-a_n'(\eta,x)]\log\frac{a_n'(\eta_x,x)}{a_n'(\eta,x)}$$

Now we can replace the first sum by twice the same sum over just those η and x for which $\eta(x)=1$, since replacing η by η_x in the summand has no effect. The required statement then follows, since the terms in the first sum vanish for $(x,\eta)\notin B_n$, as can be seen by the facts below.

Let $\eta(x)=1$, where $(x,\eta)\notin B_n$.

Case A. Here i is the distance to the closest 1 to the left of x, and j is the distance to the closest 1 to the right of x, and $i,j>1$.

Then

$$\frac{a_n'(\eta_x,x)}{M_n'(\eta_x)}=c(\eta_x,x)=\beta(i,j)=\frac{F(i+j)}{F(i)\,F(j)}$$

$$=\beta(1,1)\frac{g(i-1)\,g(1)\,g(j-1)}{g(i+j-1)}=\beta(1,1)\frac{N_n(\eta)}{N_n(\eta_x)}$$

while

$$\frac{M_n'(\eta)}{a_n'(\eta,x)}=\frac{1}{\beta(1,1)}$$

Case B. Here i is the distance to the closest 0 to the left of x, and j is the distance to the closest 0 to the right of x, and $i,j>1$.

Then

$$\frac{a_n'(\eta,x)}{M_n'(\eta)}=c(\eta,x)=\frac{F(i+j)}{F(i)\,F(j)}=\beta(1,1)\frac{N_n(\eta_x)}{N_n(\eta)}$$

while

$$\frac{M_n'(\eta_x)}{a_n'(\eta_x,x)}=\frac{1}{\beta(1,1)}$$

Case C. Here i is the distance to the closest 1 to the left of x, and j is the distance to the closest 1 to the right of x, and $i>1$, $j=1$. (We get the same basic result by assuming $i=1$, $j>1$.)

Then

$$\frac{a_n^t(\eta_x, x)}{M_n^t(\eta_x)} = c(\eta_x, x) = \beta(i, 1) = \frac{F(i+1)}{F(i)\,F(1)} = \frac{g(i-1)}{g(i)\,F(1)}\,\theta$$

Now k is the distance to the closest 0 to be left of x, and l is the distance to the closest 0 to the right of x, and $k=1$, $l>1$.

Then

$$\frac{a_n^t(\eta, x)}{M_n^t(\eta)} = c(\eta, x) = \beta(1, l) = \frac{F(l+1)}{F(l)\,F(1)} = \frac{g(l-1)}{g(l)\,F(1)}\,\theta$$

while

$$\frac{g(i-1)\,g(l)}{g(i)\,g(l-1)} = \frac{N_n(\eta)}{N_n(\eta_x)} \quad \blacksquare$$

The aim at this point is to show that for any interval $[0, T]$ the supremum over $t \in [0, T]$ of the first sum in the expression on the right of the identity in Lemma 3.1 is bounded above by something that is $o(n)$ at least along some subsequence. This will be accomplished using a series of lemmas and propositions. Let $a = \inf_n \beta(1, n) > 0$ and $b = \sup_{l,r} \beta(l, r) < \infty$. Set $G(k) = \sum_{l=k}^{\infty} g(l)$. Since $g(k)/g(k+1) \downarrow \gamma$ as $k \to \infty$, by Lemma 3.12 in Liggett[8] we have

$$\lim_{n \to \infty} \frac{G(n)}{g(n)} = \frac{\gamma}{\gamma - 1} < \infty \tag{3.2}$$

and

$$\lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} G(k)}{g(n)} = \left(\frac{\gamma}{\gamma - 1}\right)^2 < \infty \tag{3.3}$$

The first step will be to find upper bounds for $N_n(\eta)/N_n(\eta_x)$ and $N_n(\eta_x)/N_n(\eta)$ when $(x, \eta) \in B_n$.

Lemma 3.4. Let $(x, \eta) \in B_n$, where $0 < x < n$ and $\eta = 0$ on $Z_{0,x-1}$, or $x = 0$. Then $N_n(\eta)/N_n(\eta_x)$ is bounded above by a constant independent of n, x, and η. For $x > 0$, if $\eta = 0$ on $Z_{0,n} \backslash \{x\}$, then

$$\frac{N_n(\eta_x)}{N_n(\eta)} \leqslant \frac{M}{\beta(x+1, n-x+1)}$$

for some constant M independent of n, x, and η. If $\eta=0$ on $Z_{0,v}\backslash\{x\}$ and $\eta(v+1)=1$, where $0<x<v<n$, then

$$\frac{N_n(\eta_x)}{N_n(\eta)}\leqslant\frac{N}{\beta(x+1,v-x+1)}$$

for some constant N independent of n, x, and η. Otherwise, $N_n(\eta_x)/N_n(\eta)$ is bounded above by a constant independent of n, x, and η.

Proof. Case A. $0<x<n$, $\eta=0$ on $Z_{0,x-1}$, and $\eta(x+1)=0$.

1. If $\eta=0$ on $Z_{0,n}\backslash\{x\}$, then

$$N_n(\eta_x)=\sum_{\substack{u<0\\ v\geqslant n}} g(v-u)=\sum_{k=n+1}^{\infty} G(k)$$

and

$$N_n(\eta)=\sum_{\substack{u\leqslant 0\\ v\geqslant n}} g(x-u)\,g(1)\,g(v-x)=G(x)\,G(n-x)\,g(1)$$

So

$$\begin{aligned}\frac{N_n(\eta_x)}{N_n(\eta)}&=\frac{\sum_{k=n+1}^{\infty}G(k)}{g(n+1)}\,\frac{g(x)\,g(n-x)\,g(1)}{G(x)\,G(n-x)\,g(1)}\\&\quad\times\frac{F(x+1)\,F(n-x+1)\,\beta(1,1)}{F(n+2)}\\&=\frac{\sum_{k=n+1}^{\infty}G(k)}{g(n+1)}\,\frac{g(x)\,g(n-x)}{G(x)\,G(n-x)}\,\frac{\beta(1,1)}{\beta(x+1,n-x+1)}\end{aligned}$$

Thus by (3.2) and (3.3)

$$\frac{N_n(\eta_x)}{N_n(\eta)}\leqslant\frac{M}{\beta(x+1,n-x+1)}$$

and

$$\frac{N_n(\eta)}{N_n(\eta_x)}\leqslant\tilde{M}$$

for some constants M and $\tilde{M}$ which are independent of n, x, and η.

2. For $x<v<n$, if $\eta=0$ on $Z_{0,v}\backslash\{x\}$ and $\eta(v+1)=1$, then

$$N_n(\eta_x)=\sum_{u<0} g(v-u)K=G(v+1)K$$

for some constant K, and

$$N_n(\eta)=\sum_{u\leqslant 0} g(v-x)\,g(1)\,g(x-u)K=g(v-x)\,g(1)\,G(x)K$$

So

$$\frac{N_n(\eta_x)}{N_n(\eta)}=\frac{G(v+1)}{g(v+1)}\frac{g(v-x)\,g(1)\,g(x)}{g(v-x)\,g(1)\,G(x)}\frac{F(x+1)\,F(v-x+1)\,\beta(1,1)}{F(v+2)}$$

$$=\frac{G(v+1)}{g(v+1)}\frac{g(x)}{G(x)}\frac{\beta(1,1)}{\beta(x+1,v-x+1)}$$

Thus by (3.2)

$$\frac{N_n(\eta_x)}{N_n(\eta)}\leqslant\frac{N}{\beta(x+1,v-x+1)}$$

and

$$\frac{N_n(\eta)}{N_n(\eta_x)}\leqslant\tilde{N}$$

for some constants N and $\tilde{N}$ independent of n, x, and η.

Case B. $0<x<n$, $\eta=0$ on $Z_{0,x-1}$, and $\eta(x+1)=1$. Then

$$N_n(\eta_x)=\sum_{\substack{k>x\\ l}} g(k)\,g(l)\,C$$

$$=\sum_{\substack{k>x\\ l}}\frac{F(2)}{\beta(1,1)}\frac{1}{F(k+1)}\frac{1}{F(l+1)}\theta^{k+l}C$$

$$=\sum_{\substack{k>x\\ l}}\frac{F(2)}{\beta(1,1)}\frac{\beta(k+1,l+1)}{F(k+l+2)}\theta^{k+l}C$$

$$=\sum_{\substack{k>x\\ l}}\frac{F(2)}{\beta(1,1)}\frac{\beta(k,l+1)\,\beta(1,k+l+1)}{\beta(1,k)\,F(k+l+2)}\theta^{k+l}C$$

where the sum over l is either over all $l \geqslant n-x$ or just one fixed l, and C is a constant, while

$$\begin{aligned}
N_n(\eta) &= \sum_{\substack{k>x \\ l}} g(k-1)\, g(l+1)\, C \\
&= \sum_{\substack{k>x \\ l}} \frac{F(2)}{\beta(1,1)} \frac{1}{F(k)} \frac{1}{F(l+2)} \theta^{k+l} C \\
&= \sum_{\substack{k>x \\ l}} \frac{F(2)}{\beta(1,1)} \frac{\beta(k, l+2)}{F(k+l+2)} \theta^{k+l} C \\
&= \sum_{\substack{k>x \\ l}} \frac{F(2)}{\beta(1,1)} \frac{\beta(k, l+1)\, \beta(1, k+l+1)}{\beta(1, l+1)\, F(k+l+2)} \theta^{k+l} C
\end{aligned}$$

Thus

$$\frac{N_n(\eta)}{N_n(\eta_x)} \leqslant \frac{b}{a}$$

and

$$\frac{N_n(\eta_x)}{N_n(\eta)} \leqslant \frac{b}{a}$$

Case C. $x=0$.

1. For $0<u<n$, if $\eta=0$ on $Z_{0,u} \backslash \{x\}$ and $\eta(u+1)=1$, then

$$N_n(\eta) = G(1)\, g(u)\, K'$$

and

$$N_n(\eta_x) = G(u+1)\, K'$$

for some constant K'. Hence

$$\frac{N_n(\eta)}{N_n(\eta_x)} = \frac{G(1)\, g(u)}{G(u+1)} = \frac{G(1)\, g(u+1)\, g(u)}{G(u+1)\, g(u+1)} \leqslant M'$$

for some constant M' independent of n, x, and η. Also

$$\frac{N_n(\eta_x)}{N_n(\eta)} = \frac{G(u+1)}{G(1)\, g(u)} \leqslant \frac{G(u)}{G(1)\, g(u)} \leqslant M''$$

for some constant M'' independent of n, x, and η.

2. If $\eta = 0$ on $Z_{0,n} \backslash \{x\}$, then

$$N_n(\eta_x) = \sum_{\substack{u<0\\ v \geqslant n}} g(u-v) = \sum_{k=n+1}^{\infty} G(k)$$

and

$$N_n(\eta) = G(1) \sum_{u \geqslant n} g(u) = G(1)\, G(n)$$

Thus

$$\begin{aligned}
\frac{N_n(\eta)}{N_n(\eta_x)} &= \frac{G(1)\, G(n)}{\sum_{k=n+1}^{\infty} G(k)} \\
&= \frac{G(1)[\,g(n) + G(n+1)]}{\sum_{k=n+1}^{\infty} G(k)} \\
&= \frac{G(1)\, g(n)}{\sum_{k=n+1}^{\infty} G(k)} + \frac{G(1)\, G(n+1)}{\sum_{k=n+1}^{\infty} G(k)} \\
&\leqslant \frac{G(1)\, g(n)}{g(n+1)} + \frac{G(1)\, G(n+1)}{\sum_{k=n+1}^{\infty} G(k)} \leqslant N'
\end{aligned}$$

for some constant N' independent of n, x, and η. Also

$$\frac{N_n(\eta_x)}{N_n(\eta)} = \frac{\sum_{k=n+1}^{\infty} G(k)}{G(1)\, G(n)} \leqslant \frac{\sum_{k=n+1}^{\infty} G(k)}{G(1)\, g(n+1)} \leqslant N''$$

for some constant N'' independent of n, x, and η.

3. For $0 < u < n$, if $\eta = 1$ on $Z_{0,u}$ and $\eta(u+1) = 0$, then

$$N_n(\eta) = \sum_{l > u} g(l)\, K'' = G(u+1)\, K''$$

and

$$N_n(\eta_x) = G(1)\, g(u)\, K''$$

for some constant K''. Thus, by basically the same estimates as are used in 1, $N_n(\eta)/N_n(\eta_x)$ and $N_n(\eta_x)/N_n(\eta)$ are both bounded by constants independent of n, x, and η.

4. For $\eta = 1$ on $Z_{0,n}$ use the same ideas as in 2. ∎

Remark. Note that if we use the facts that the function ν is symmetric in zeros and ones and the role of left and right can be interchanged, then Lemma 3.4 covers all possibilities, and hence we have upper bounds on $N_n(\eta)/N_n(\eta_x)$ and $N_n(\eta_x)/N_n(\eta)$ for all $(x, \eta) \in B_n$.

Lemma 3.5. There exists a positive constant c so that

$$\beta(k, l) \geqslant cg(k)\, \gamma^k$$

for all k and l.

Proof. If $k, l > 1$, then

$$\begin{aligned}\beta(k, l) &= \frac{\beta(1, 1)\, g(k-1)\, g(1)\, g(l-1)}{g(k+l-1)} \\ &= \beta(1, 1)\, g(1) \frac{g(k-1)}{g(k)}\, g(k) \frac{g(l-1)}{g(k+l-1)} \\ &\geqslant cg(k)\, \gamma^k\end{aligned}$$

for some positive constant c, because

$$\frac{g(n)}{g(n+k)} \geqslant \gamma^k c' \qquad \forall n, k \geqslant 1$$

for some $c' > 0$, since

$$\frac{g(n)}{g(n+1)} \geqslant \gamma \qquad \forall n \geqslant N$$

The result follows since $\inf_{k \geqslant 1} \beta(k, 1) > 0$ and $g(k)\, \gamma^k$ is bounded. ∎

Let

$$h_t(k) = \mu_t\{\zeta \in X: \zeta(0) = 1, \zeta(k) = 1, \text{and } \zeta = 0 \text{ on } Z_{1,k-1}\}$$

and

$$H_t(k) = \sum_{l=k}^{\infty} h_t(l)$$

Then $\sum_{k=1}^{\infty} H_t(k) < \infty$, since μ_t is translation invariant and concentrates on $\tilde{X}$.

Lemma 3.6. If $\eta_x = 0$ on $Z_{0,n}$, then

$$\frac{M_n^t(\eta_x)}{a_n^t(\eta_x, x)} \leqslant \frac{\sum_{k \geqslant n+2} H_t(k)}{cg(n+1)\, \gamma^{n+1} H_t(n+2)}$$

and for $0 \leqslant x < v < n$, if $\eta_x = 0$ on $Z_{0,v}$ and $\eta(v+1) = 1$, then

$$\frac{M_n^t(\eta_x)}{a_n^t(\eta_x, x)} \leqslant \frac{1}{cg(n+1)\,\gamma^{n+1}}$$

for some constant $c > 0$ independent of n, x, and η. Similarly, if $\eta = 1$ on $Z_{0,n}$, then

$$\frac{M_n^t}{a_n^t(\eta, x)} \leqslant \frac{\sum_{k \geqslant n+2} H_t(k)}{cg(n+1)\,\gamma^{n+1} H_t(n+2)}$$

and if $\eta = 1$ on $Z_{0,v}$ and $\eta(v+1) = 0$, then

$$\frac{M_n^t(\eta)}{a_n^t(\eta, x)} \leqslant \frac{1}{cg(n+1)\,\gamma^{n+1}}$$

Proof. If $\eta_x = 0$ on $Z_{0,n}$, then using the previous lemma and the fact that $g(k)\,\gamma^k$ is decreasing for large k,

$$\begin{aligned}
a_n^t(\eta_x, x) &= \sum_{\substack{u<0\\ v>n}} \beta(x-u, v-x)\, h_t(v-u) \\
&\geqslant \sum_{\substack{u<0\\ v>n}} cg(x-u)\,\gamma^{x-u} h_t(v-u) \\
&\geqslant cg(x+1)\,\gamma^{x+1} \sum_{v>n} h_t(v+1) \\
&\geqslant cg(n+1)\,\gamma^{n+1} H_t(n+2)
\end{aligned}$$

for some positive constant c independent of n, x, and η, while

$$M_n^t(\eta_x) = \sum_{\substack{u<0\\ v>n}} h_t(v-u) = \sum_{k \geqslant n+2} H_t(k)$$

If $\eta_x = 0$ on $Z_{0,v}$ and $\eta(v+1) = 1$, then

$$\begin{aligned}
a_n^t(\eta_x, x) = &\sum_{u<0} \beta(x-u, v-x+1) \\
&\times \mu_t\{\zeta\colon \zeta(u) = 1, \zeta = 0 \text{ on } Z_{u+1,0}, \text{ and } \zeta = \eta_x \text{ on } Z_{0,n}\} \\
\leqslant\, & cg(n+1)\,\gamma^{n+1} \\
&\times \sum_{u<0} \mu_t\{\zeta\colon \zeta(u) = 1, \zeta = 0 \text{ on } Z_{u+1,0}, \text{ and } \zeta = \eta_x \text{ on } Z_{0,n}\}
\end{aligned}$$

while

$$M_n^t(\eta_x) = \sum_{u<0} \mu_t\{\zeta : \zeta(u) = 1, \zeta = 0 \text{ on } Z_{u+1,0}, \text{ and } \zeta = \eta_x \text{ on } Z_{0,n}\}$$

The second statement follows from the first and the fact that μ_t is symmetric in zeros and ones. ∎

Lemma 3.7. We have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \frac{1}{n} \sum_{(x,\eta)\in B_n} a_n^t(\eta_x, x) = 0$$

and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \frac{1}{n} \sum_{(x,\eta)\in B_n} a_n^t(\eta, x) = 0$$

Proof. For all $t \in [0, T]$

$$\sum_{(x,\eta)\in B_n} M_n^t(\eta_x) = \sum_{x=0}^{n} \sum_{\eta:\,(x,\eta)\in B_n} M_n^t(\eta_x)$$

$$\leqslant 2 \sum_{x=0}^{n} [\mu_t\{\zeta : \zeta = 0 \text{ on } Z_{0,x-1}\} + \mu_t\{\zeta : \zeta = 1 \text{ on } Z_{0,x-1}\}]$$

$$= 4 \sum_{x=0}^{n} \int (1 - e^{-bT})^{\sum_{i=1}^{x} \eta(i)}\, d\mu$$

So

$$\sup_{t\in[0,T]} \frac{1}{n} \sum_{(x,\eta)\in B_n} M_n^t(\eta_x) \to 0$$

by the fact that μ concentrates on $\tilde{X}$ and the dominated convergence theorem. The first statement follows since $a_n^t(\eta_x, x) \leqslant bM_n^t(\eta_x)$. The proof of the second statement is essentially the same. ∎

With the preceding technical lemmas in hand, we now can show that the previously mentioned quantity is $o(n)$ by dividing B_n into disjoint sets corresponding to bounds we have obtained for various terms.

Proposition 3.8. Let

$$\tilde{B}_n = \{(x, \eta) \in B_n : 0 < x < n, \eta(x-1) = 0 \text{ and } \eta(x+1) = 1, \text{ or } \eta(x-1) = 1 \text{ and } \eta(x+1) = 0; \text{ or } x = 0 \text{ and } \eta(x+1) = 0, \text{ or } x = n \text{ and } \eta(x-1) = 0\}$$

Let

$$\tilde{\tilde{B}}_n = \{(x, \eta) \in B_n \colon 0 < x < n,\ \eta(x-1) = 0 \text{ and } \eta(x+1) = 1,$$
$$\text{or } \eta(x-1) = 1 \text{ and } \eta(x+1) = 0; \text{ or } x = 0 \text{ and}$$
$$\eta(x+1) = 1, \text{ or } x = n \text{ and } \eta(x-1) = 1\}$$

Then

$$\limsup_n \sup_{t \in [0,T]} \frac{1}{n} \sum_{(x,\eta) \in \bar{B}_n} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \leqslant 0$$

and

$$\limsup_n \sup_{t \in [0,T]} \frac{1}{n} \sum_{(x,\eta) \in \tilde{\tilde{B}}_n} -a_n^t(\eta, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \leqslant 0$$

Proof. On $\tilde{B}_n$

$$\frac{a_n^t(\eta_x, x)}{M_n^t(\eta_x)} \leqslant b, \qquad \frac{M_n^t(\eta)}{a_n^t(\eta, x)} \leqslant \frac{1}{a}$$

and $N_n(\eta_x)/N_n(\eta)$ is bounded by a constant. Thus

$$\limsup_n \sup_{t \in [0,T]} \frac{1}{n} \sum_{(x,\eta) \in \tilde{B}_n} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)}$$
$$\leqslant K \limsup_n \sup_{t \in [0,T]} \frac{1}{n} \sum_{(x,\eta) \in \tilde{B}_n} a_n^t(\eta_x, x) = 0$$

for some constant $K > 0$. The proof of the second statement is basically the same. ∎

Proposition 3.9. Let

$$\hat{B}_n = \{(x, \eta) \in B_n \colon \eta = 1 \text{ on } Z_{0,n}\}$$

and

$$\hat{\hat{B}}_n = \{(x, \eta) \in B_n \colon \eta_x = 0 \text{ on } Z_{0,n}\}$$

Then

$$\liminf_n \left[\frac{1}{n} \sum_{(x,\eta) \in \hat{B}_n} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \right.$$
$$\left. - \frac{1}{n} \sum_{(x,\eta) \in \hat{\hat{B}}_n} a_n^t(\eta, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \right] \leqslant 0$$

While if there exists a $\delta > 0$ such that

$$\mu\{\eta(0)=0 \mid \eta(x)=0 \text{ for all } 0<x<n \text{ and } \eta(n)=1\} \geqslant \delta \tag{3.10}$$

for all $n \geqslant 1$, then

$$\limsup_n \sup_{t\in[0,T]} \left[\frac{1}{n} \sum_{(x,\eta)\in \hat{B}_n} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \right.$$
$$\left. -\frac{1}{n} \sum_{(x,\eta)\in \hat{\hat{B}}_n} a_n^t(\eta, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)}\right] \leqslant 0$$

Proof. On $\hat{B}_n$

$$\frac{a_n^t(\eta_x, x)}{M_n^t(\eta_x)} \leqslant b, \qquad \frac{M_n^t(\eta)}{a_n^t(\eta, x)} \leqslant \frac{\sum_{k\geqslant n+2} H_t(k)}{cg(n+1)\gamma^{n+1} H(n+2)}$$

and $N_n(\eta_x)/N_n(\eta)$ is bounded by a constant. On $\hat{\hat{B}}_n$

$$\frac{a_n^t(\eta, x)}{M_n^t(\eta)}, \qquad \frac{M_n^t(\eta_x)}{a_n^t(\eta_x, x)}, \qquad \frac{N_n(\eta)}{N_n(\eta_x)}$$

are bounded by the same things, respectively. Since

$$\sum_{(x,\eta)\in \hat{B}_n} a_n^t(\eta_x, x)$$

and

$$\sum_{(x,\eta)\in \hat{\hat{B}}_n} a_n^t(\eta, x)$$

are bounded, it suffices to show that

$$\liminf_n \frac{1}{n} \log\left(\frac{\sum_{k\geqslant n+2} H_t(k)}{g(n+1)\,\gamma^{n+1} H_t(n+2)}\right) \leqslant 0$$

to prove our first claim. Writing

$$\frac{1}{n}\log\left(\frac{\sum_{k\geqslant n+2} H_t(k)}{g(n+1)\,\gamma^{n+1} H_t(n+2)}\right)$$
$$=\frac{1}{n}\log\frac{\sum_{k\geqslant n+2} H_t(k)}{H_t(n+2)} - \frac{\log g(n+1)}{n} - \frac{n+1}{n}\log\gamma$$

our result follows since

$$\lim_n \frac{1}{n} \log g(n) = -\log \gamma$$

and

$$\liminf_n \frac{1}{n} \log \frac{\sum_{k \geqslant n} H_t(k)}{H_t(n)} = 0$$

by Lemmas 3.7 and 3.8, respectively, in Liggett.[8] Now

$$\sum_{(x,\eta) \in \tilde{B}_n} a_n^t(\eta_x, x)$$

and

$$\sum_{(x,\eta) \in \hat{B}_n} a_n^t(\eta, x)$$

are both bounded above by

$$\begin{aligned} &b\mu_t\{\eta: \eta(x) = 0 \text{ for } n \text{ sites } x \in Z_{0,n}\} \\ &\leqslant 2b\mu_t\{\eta: \eta = 0 \text{ on } Z_{0,[n/2]}\} \\ &\leqslant 2b \int (1 - e^{-bT})^{\sum_{i=1}^{[n/2]} \eta(i)}\, d\mu \to 0 \end{aligned}$$

since μ concentrates on $\tilde{X}$. So to prove our result in the case where μ satisfies (3.10) it suffices to show

$$\sup_{t \in [0,T]} \frac{1}{n} \log \left(\frac{\sum_{k \geqslant n+2} H_t(k)}{g(n+1)\, \gamma^{n+1}\, H_t(n+2)} \right)$$

is bounded above in n. This follows again by Liggett's Lemma 3.7 and the fact that

$$\sup_{t \in [0,T]} \frac{1}{n} \log \left(\frac{1}{H_t(n+2)} \right) \leqslant \frac{1}{n} \log \left(\frac{e^{(n+2)bT}}{H_0(n+2)} \right)$$

where

$$\frac{1}{n} \log \left(\frac{1}{H_0(n+2)} \right) = \frac{1}{n \log H_0(2)} - \frac{1}{n} \sum_{k=1}^{n} \log \frac{H_0(k+2)}{H_0(k+1)}$$

is bounded above by (3.10).

Proposition 3.11. Let

$$B_n' = \{(x,\eta) \in B_n\colon 0 < x < n, \eta(x-1) = 1 \text{ and } \eta(x+1) = 1;$$
$$\text{or } x = 0 \text{ and } \eta(x+1) = 1, \text{ or } x = n \text{ and } \eta(x-1) = 1\} \backslash \hat{B}_n$$

Let

$$B_n'' = \{(x,\eta) \in B_n\colon 0 < x < n, \eta(x-1) = 0 \text{ and } \eta(x+1) = 0;$$
$$\text{or } x = 0 \text{ and } \eta(x+1) = 0, \text{ or } x = n \text{ and } \eta(x-1) = 0\} \backslash \hat{\hat{B}}_n$$

Then

$$\limsup_n \sup_{t \in [0,T]} \left[\frac{1}{n} \sum_{(x,\eta) \in B_n'} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \right.$$
$$\left. - \frac{1}{n} \sum_{(x,\eta) \in B_n''} a_n^t(\eta, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \right] \leqslant 0$$

Proof. On B_n'

$$\frac{a_n^t(\eta_x, x)}{M_n^t(\eta_x)} \leqslant b, \qquad \frac{M_n^t(\eta)}{a_n^t(\eta, x)} \leqslant \frac{1}{cg(n+1)\,\gamma^{n+1}}$$

and $N_n(\eta_x)/N_n(\eta)$ is bounded by a constant. On B_n''

$$\frac{a_n^t(\eta, x)}{M_n^t(\eta)}, \qquad \frac{M_n^t(\eta_x)}{a_n^t(\eta_x, x)}, \qquad \frac{N_n(\eta)}{N_n(\eta_x)}$$

are bounded by the same things, respectively. So as in the previous proposition it suffices to prove that

$$\frac{1}{n} \log \left(\frac{1}{g(n+1)\,\gamma^{n+1}} \right) = -\frac{\log g(n+1)}{n} - \frac{n+1}{n} \log \gamma$$

converges to 0. This follows as above. ∎

Proposition 3.12. Let

$$B_n^* = \{(x,\eta) \in B_n\colon 0 < x < n, \eta(x-1) = 0, \text{ and } \eta(x+1) = 0\}$$

Let

$$B_n^{**} = \{(x,\eta) \in B_n\colon 0 < x < n, \eta(x-1) = 1, \text{ and } \eta(x+1) = 1\}$$

Then

$$\limsup_n \sup_{t\in[0,T]} \frac{1}{n} \sum_{(x,\eta)\in B_n^*} a_n^t(\eta_x, x) \log \frac{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)}{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)} \leqslant 0$$

and

$$\limsup_n \sup_{t\in[0,T]} \frac{1}{n} \sum_{(x,\eta)\in B_n^{**}} a_n^t(\eta, x) \log \frac{a_n^t(\eta, x)\, M_n^t(\eta_x)\, N_n(\eta)}{a_n^t(\eta_x, x)\, M_n^t(\eta)\, N_n(\eta_x)} \leqslant 0$$

Proof. On B_n^*

$$\frac{M_n^t(\eta)}{a_n^t(\eta, x)} = \frac{1}{\beta(1,1)} \qquad \text{and} \qquad \frac{a_n^t(\eta_x, x)}{M_n^t(\eta_x)} \leqslant b$$

Also:

Case A. $\eta_x = 0$ on $Z_{0,n}$:

$$\frac{N_n(\eta_x)}{N_n(\eta)} \leqslant \frac{M}{\beta(x+1, n-x+1)}$$

Case B. $0 < x < v < n$, $\eta_x = 0$ on $Z_{0,v}$ and $\eta(v+1) = 1$:

$$\frac{N_n(\eta_x)}{N_n(\eta)} \leqslant \frac{N}{\beta(x+1, v-x+1)}$$

by Lemma 3.4. So to prove the first claim of this proposition in Cases A and B by the monotonicity of $\beta(l, r)$ it suffices to show

$$\frac{1}{n} \sum_{x=0}^{n} \beta(x+1, n-x+1) \times \mu_t\{\zeta : \zeta = 0 \text{ on } Z_{0,n}\}\ |\log \beta(x+1, n-x+1)|$$

and

$$\frac{1}{n} \sum_{x=1}^{n-1} \sum_{v=x+1}^{n-1} \beta(x+1, v-x+1) \times \mu_t\{\zeta : \zeta = 0 \text{ on } Z_{0,v}, \zeta(v+1) = 1\}\ |\log \beta(x+1, v-x+1)|$$

both converge to zero uniformly on $[0, T]$. The function $s \log s$ is bounded on compact subsets of $[0, \infty)$ and $\beta(l, r)$ is uniformly bounded, hence these statements reduce to

$$\lim_n \sup_{t \in [0,T]} \mu_t\{\zeta ; \zeta = 0 \text{ on } Z_{0,n}\}$$

$$\leqslant \lim_n \int (1 - e^{-bT})^{\sum_{i=1}^n \eta(i)}\, d\mu = 0$$

which is true since μ concentrates on $\tilde{X}$. The second claim follows by symmetry. ∎

Now

$$\tilde{B}_n \cup \hat{B}_n \cup B'_n \cup B^*_n = \tilde{\tilde{B}}_n \cup \hat{\hat{B}}_n \cup B''_n \cup B^{**}_n = B_n \tag{3.13}$$

Set

$$C'_n(x, \eta) = [a'_n(\eta_x, x) - a'_n(\eta, x)] \log \frac{a'_n(\eta_x, x)\, M'_n(\eta)\, N_n(\eta_x)}{a'_n(\eta, x)\, M'_n(\eta_x)\, N_n(\eta)}$$

and set

$$D'_n(x) = \sum_{\eta \in X_{0,n}} [a'_n(\eta_x, x) - a'_n(\eta, x)] \log \frac{a'_n(\eta_x, x)}{a'_n(\eta, x)}$$

for $x \in Z_{0,n}$ and $\eta \in X_{0,n}$. Notice that $D'_n(x) \geqslant 0$ for all x and n.

Corollary 3.14. If μ is invariant, then

$$\liminf_n \frac{1}{n} \sum_{x=0}^{n} D^0_n(x) = \liminf_n \frac{1}{n} \sum_{(x,\eta) \in B_n} C^0_n(x, \eta) = 0$$

while if μ satisfies (3.10), then

$$\limsup_n \sup_{t \in [0,T]} \frac{1}{n} \sum_{(x,\eta) \in B_n} C^t_n(x, \eta) \leqslant 0$$

Proof. This follows immediately from Lemma 3.1, Propositions 3.8, 3.9, 3.11, and 3.12, and the identity (3.13). ∎

Proposition 3.15. Any symmetric renewal measure is ergodic with respect to translation.

Proof. Let μ_β be the symmetric renewal measure corresponding to the probability density β. Set $T\colon X \to X$ to be the transformation which corresponds to translation by one unit to the right. Since the collection $\mathfrak{D}$

of functions which depend on only finitely many coordinates is dense in $C(X)$, to prove that μ_β is ergodic it suffices to check that

$$\int f(T^{-n})\, g\, d\mu_\beta \to \int f\, d\mu_\beta \int g\, d\mu_\beta \tag{3.16}$$

as $n \to \infty$ for all $f, g \in \mathfrak{D}$. It is enough to check (3.16) for f and g which are indicator functions of cylinder sets, since any element of $\mathfrak{D}$ is a finite linear combination of such functions. Let $\varepsilon > 0$. Choose M so that

$$\mu_\beta(\overbrace{0 \cdots 0}^{M}) < \varepsilon$$

This is possible because $\mu_\beta(\tilde{X}) = 1$. We will identify each cylinder set with a finite number of specified coordinates. Suppose A and B are cylinder sets. Let n be large enough so that the distance between coordinates specified by $T^{-n}A$ and those specified by B is greater than $2M$. Assume (wlog) the last coordinate specified by A is a 0, and the first specified by B is a 1. Then

$$\left| \mu_\beta(T^{-n}A \overbrace{\cdots}^{>2M} B) - \sum_{k,l=0}^{M-1} \mu_\beta(T^{-n}A \overbrace{\cdots 0}^{k} \underbrace{1 \cdots}_{m} \overbrace{1 \cdots}^{l} B) \right| < 2\varepsilon$$

For $0 \leqslant k,\ l < M$, we have

$$\mu_\beta(T^{-n}A \overbrace{\cdots 0}^{k} \underbrace{1 \cdots 0}_{m} \overbrace{1 \cdots}^{l} B) = \frac{\mu_\beta(T^{-n}A \overbrace{\cdots 0}^{k} 1)}{\mu_\beta(0\,1)} \frac{\mu_\beta(0 \overbrace{1 \cdots}^{l} B)}{\mu_\beta(0\,1)} \mu_\beta(0 \overbrace{1 \cdots 0}^{m} 1)$$

From the proof of Proposition 2.1 we know that

$$\mu_\beta(0 \overbrace{1 \cdots 0}^{m} 1) \to \mu_\beta(0\,1)\, \mu_\beta(0\,1) \qquad \text{as} \quad m \to \infty$$

Since $m \to \infty$ as $n \to \infty$ for fixed k and l, we can choose n so large that

$$\left| \frac{\mu_\beta(T^{-n}A \overbrace{\cdots 0}^{k} 1)}{\mu_\beta(0\,1)} \frac{\mu_\beta(0 \overbrace{1 \cdots}^{l} B)}{\mu_\beta(0\,1)} \mu_\beta(0 \overbrace{1 \cdots 0}^{m} 1) - \mu_\beta(T^{-n}A \overbrace{\cdots 0}^{k} 1)\, \mu_\beta(0 \overbrace{1 \cdots}^{l} B) \right|$$

$$< \frac{\varepsilon}{M^2}$$

for all $0 \leqslant k,\ l < M$. Using the fact that

$$\left|\sum_{k,l=0}^{M-1} \mu_\beta(T^{-n}A\overbrace{\cdots 0}^{k}1)\,\mu_\beta(0\,\overbrace{1\cdots}^{l}B) - \mu_\beta(T^{-n}A)\,\mu_\beta(B)\right| < 2\varepsilon$$

we get (3.16) for the indicator functions 1_A and 1_B. ∎

Proof of Theorem 1.4. Let μ be any translation-invariant, invariant measure on $\tilde{X}$ which is symmetric in zeros and ones. It follows from Corollary 3.14 and a standard "free energy" argument (see Liggett[8]) that

$$a_n(\eta_x, x) = a_n(\eta, x)$$

for all $n \geqslant 0$, $x \in Z_{0,n}$, and $\eta \in X_{0,n}$. Thus μ is reversible. Now suppose we were not given that μ is symmetric in zeros and ones. Then consider the measure $\tilde{\mu}$ which is defined by applying μ to configurations with zeros and ones interchanged. The measure $\frac{1}{2}\mu + \frac{1}{2}\tilde{\mu}$ is then symmetric in zeros and ones, and hence reversible. It is also a symmetric renewal measure. Since an ergodic measure is an extremal translation-invariant measure (see Corollary 4.14 in Chapter 1 of Liggett[9]), we get $\mu = \tilde{\mu}$. Thus our proof is complete. ∎

In the attractive case we can extend the infinite nearest-particle process to include the starting configurations $\eta(x) \equiv 0$ and $\eta(x) \equiv 1$. For $g \in \mathfrak{D}$, Ωg extends continuously to X, since by attractiveness $\beta(l, r)$ extends continuously to a function on $\{1, 2,..., \infty\} \times \{1, 2,..., \infty\}$. Let g be an increasing continuous function on X. If $\eta, \zeta \in \tilde{X}$ and $\eta \leqslant \zeta$, then

$$S(t)\,g(\eta) \leqslant S(t)\,g(\zeta)$$

and so we can define

$$S(t)\,g(\bar{1}) = \lim_{\substack{\eta \in \tilde{X} \\ \eta \uparrow \bar{1}}} S(t)\,g(\eta)$$

where $\bar{1}$ is the configuration $\zeta(x) \equiv 1$, and we can define $S(t)\,g(\bar{0})$ similarly. Thus $S(t)\,g$ extends continuously to $\bar{0}$ and $\bar{1}$ for $g \in C(X)$. Set $\alpha(n) = \sum_{l+r=n+1} (l \wedge r)\,\beta(l, r)$, and let $\tilde{\alpha}(n) = \max_{k \leqslant n} \alpha(k)$. By basically the same arguments as in Liggett,[8] if

$$\sum_n \frac{1}{\alpha(n)} < \infty \tag{3.17}$$

then δ_0 and δ_1 are not invariant, while if

$$\sum_n \frac{1}{\tilde{\alpha}(n)} = \infty \tag{3.18}$$

they are invariant. Thus if $\beta(l, r)$ satisfy (1.2) and (3.17), then by Theorem 1.4 the renewal measure is the only translation-invariant, invariant measure, and the process is ergodic, while if $\beta(l, r)$ satisfy (1.2) and (3.18), then if no reversible (renewal) measure exists the only translation-invariant, invariant measures are of the form $\lambda\delta_0 + (1-\lambda)\delta_1$, where $\lambda \in [0, 1]$.

4. WEAK LIMITS

Throughout this section we will be assuming that the rates $\beta(l, r)$ satisfy the hypothesis of Theorem 1.4. When the rates are attractive we often extend the definition of a reversible measure by saying that any translation-invariant measure μ is reversible if

$$\int_{\{\zeta:\, \zeta(y) = \eta(y) \text{ for } 0 \leqslant y \leqslant n\}} c(x, \zeta)\, d\mu(\zeta) = \int_{\{\zeta:\, \zeta(y) = \eta_x(y) \text{ for } 0 \leqslant y \leqslant n\}} c(x, \zeta)\, d\mu(\zeta)$$

for all $n \geqslant 0$, $x \in \mathbb{Z}$, and $\eta \in X$. (Note that with this definition the pointmasses on the identically zero and identically one configurations, δ_0 and δ_1, are reversible.) We will use the free energy technique and draw from many of the results obtained in Section 3 to prove the following:

Theorem 4.1. Assume that $\beta(l, r)$ satisfies (1.2), where F is positive, and that there exists a positive integer N such that $\beta(l, r)$ is monotone decreasing in l and r for $l + r \geqslant N$. Let μ be a translation-invariant probability measure on $\tilde{X}$, and suppose that there exists a $\delta > 0$ such that

$$\mu\{\eta(0) = 0 \mid \eta(x) = 0 \text{ for all } 0 < x < n \text{ and } \eta(n) = 1\} \geqslant \delta \tag{4.2}$$

and

$$\mu\{\eta(0) = 1 \mid \eta(x) = 1 \text{ for all } 0 < x < n \text{ and } \eta(n) = 0\} \geqslant \delta \tag{4.3}$$

for all $n \geqslant 1$. Let $\{t_k\}$ be a sequence of nonnegative numbers such that $t_k \to \infty$. Suppose that

$$\nu = \lim_k \mu S(t_k)$$

Then if ν concentrates on $\tilde{X}$ or the rates are attractive, ν must be a reversible measure.

We need a few propositions to obtain the proof of Theorem 4.1. Our first result uses subadditivity to show that $\lim_n[H_n(\mu)/n]$ exists. We call this $A(\mu)$, the free energy of μ.

Proposition 4.4. Let μ be a translation-invariant measure on X. Then

$$\lim_{n\to\infty}\frac{H_n(\mu)}{n}=A(\mu)$$

exists. Moreover, there exists a constant $c>-\infty$ independent of μ such that

$$A(\mu)>c$$

Proof. The hard part of the proof is in showing that if we add a constant to

$$\sum_{\eta\in X_{0,n}} M_n(\eta)\log\frac{1}{N_n(\eta)}$$

we get a subadditive sequence. We will start by showing that there exists a $K<\infty$ such that

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}\leqslant K \tag{4.5}$$

for all m, $n\geqslant 1$, where $\eta_m\in X_{0,m}$, $\eta_{n-1}\in X_{0,n-1}$, and $\eta_m\times\eta_{n-1}\in X_{0,m+n}$ such that $\eta_m\times\eta_{n-1}=\eta_m$ on $Z_{0,m}$ and $\eta_m\times\eta_{n-1}=\eta_{n-1}$ on $Z_{m+1,m+n}$. Set $G(k)=\sum_{l=k}^{\infty}g(l)$. As in Liggett,[8] we have

$$\lim_{n\to\infty}\frac{G(n)}{g(n)}=\frac{\gamma}{\gamma-1}<\infty \tag{4.6}$$

and

$$\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}G(k)}{g(n)}=\left(\frac{\gamma}{\gamma-1}\right)^2<\infty \tag{4.7}$$

Case A. $\eta_m\times\eta_{n-1}$ is a configuration of the form

$$\cdots 0\,\overbrace{1\cdots\underset{\text{site } m}{1}}^{k_1}\,\overbrace{0\cdots 0}^{k_2}\,1\cdots,\qquad k_1,k_2\geqslant 1$$

Then

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}=\frac{G(k_1)\,G(k_2)}{g(k_1)\,g(k_2)}$$

Case B. $\eta_m \times \eta_{n-1}$ is a configuration of the form

$$\cdots 0\,\overbrace{1\cdots\underset{m}{1}}^{k}\,\overbrace{0\cdots\ \underset{m+n}{0}}^{n},\qquad k\geqslant 1$$

Then

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}=\frac{G(k_1)\sum_{l\geqslant k_2}G(l)}{g(k_1)\,G(k_2)}$$

Case C. $\eta_m \times \eta_{n-1}$ is a configuration of the form

$$\overbrace{1\cdots\underset{m}{1}}^{m+1}\,\overbrace{0\cdots\ \underset{m+n}{0}}^{n}$$

Then

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}=\frac{\sum_{l\geqslant m+1}G(l)\sum_{l\geqslant n}G(l)}{G(m+1)\;G(n)}$$

Case D. $\eta_m \times \eta_{n-1}$ is a configuration of the form

$$\cdots 1\,\overbrace{0\cdots\underset{m}{0}}^{k_1}\,\overbrace{0\cdots 0}^{k_2}\,1\cdots,\qquad k_1, k_2\geqslant 1$$

Then

$$\begin{aligned}\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}&=\frac{G(k_1)\,G(k_2)}{g(k_1+k_2)}\\&=\frac{g(k_1)\,g(k_2)}{g(k_1+k_2)}\,\frac{G(k_1)\,G(k_2)}{g(k_1)\,g(k_2)}\\&\leqslant c\beta(k_1+1, k_2)\,\frac{G(k_1)\,G(k_2)}{g(k_1)\,g(k_2)}\end{aligned}$$

for some finite constant c, independent of k_i, m, and n.

Case E. $\eta_m \times \eta_{n-1}$ is a configuration of the form

$$\cdots 1\,\overbrace{0\cdots\underset{m}{0}}^{k}\,\overbrace{0\cdots\ \underset{m+n}{0}}^{n},\qquad k\geqslant 1$$

Then

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}=\frac{G(k)\sum_{l\geqslant n}G(l)}{G(k+n)}$$
$$\leqslant\frac{G(k)\sum_{l\geqslant n}G(l)}{g(k+n)}$$
$$=\frac{g(k)\,g(n)}{g(k+n)}\,\frac{G(k)\sum_{l\geqslant n}G(l)}{g(k)\,g(n)}$$
$$\leqslant c\beta(k+1,n)\,\frac{G(k)\sum_{l\geqslant n}G(l)}{g(k)\,g(n)}$$

Case F. $\eta_m\times\eta_{n-1}$ is a configuration of the form

$$\overbrace{\underset{}{0}\cdots\underset{m}{0}}^{m+1}\;\overbrace{0\cdots\;\underset{m+n}{0}}^{n}$$

Then

$$\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}=\frac{\sum_{l\geqslant m+1}G(l)\sum_{l\leqslant n}G(l)}{\sum_{l\geqslant m+n+1}G(l)}$$
$$\leqslant\frac{\sum_{l\geqslant m+1}G(l)\sum_{l\geqslant n}G(l)}{g(m+n+1)}$$
$$=\frac{g(m+1)\,g(n)}{g(m+n+1)}\,\frac{\sum_{l\geqslant m+1}G(l)}{g(m+1)}\,\frac{\sum_{l\geqslant n}G(l)}{g(n)}$$
$$\leqslant c\beta(m+1,n+1)\,\frac{\sum_{l\geqslant m+1}G(l)}{g(m+1)}\,\frac{\sum_{l\geqslant n}G(l)}{g(n)}$$

Thus (4.5) is a result of (4.6) and (4.7). Now for $m, n\geqslant 1$

$$\sum_{\eta\in X_{0,m+n}}M_{m+n}(\eta)\log\frac{1}{N_{m+n}(\eta)}-\sum_{\eta_m\in X_{0,m}}M_m(\eta_m)\log\frac{1}{N_m(\eta_m)}$$
$$-\sum_{\eta_{n-1}\in X_{0,n-1}}M_{n-1}(\eta_{n-1})\log\frac{1}{N_{n-1}(\eta_{n-1})}$$
$$=\sum_{\eta_m\in X_m}\sum_{\eta_{n-1}\in X_{n-1}}M_{m+n}(\eta_m\times\eta_{n-1})\log\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}$$
$$\leqslant\sum_{\eta_m\in X_m}\sum_{\eta_{n-1}\in X_{n-1}}M_{m+n}(\eta_m\times\eta_{n-1})\left[\frac{N_m(\eta_m)\,N_{n-1}(\eta_{n-1})}{N_{m+n}(\eta_m\times\eta_{n-1})}-1\right]$$
$$\leqslant\sum_{\eta_m\in X_m}\sum_{\eta_{n-1}\in X_{n-1}}M_{m+n}(\eta_m\times\eta_{n-1})[K-1]=K-1$$

where the first inequality is a consequence of the fact that $\log(1/t) \leqslant (1/t) - 1$, for $t > 0$, which follows from the convexity of the function $t \log t$. So if we set

$$S(n) = \sum_{\eta \in X_{0,n}} M_n(\eta) \log \frac{1}{N_n(\eta)} + K - 1$$

the sequence $S(n)$ is subadditive. Thus

$$\lim_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log \frac{1}{N_n(\eta)}$$

exists. By a similar argument (see Ruelle,[11] pp. 178–181, for example)

$$\lim_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log M_n(\eta)$$

also exists. Notice that if $\sum_{k \geqslant 1} G(k) \leqslant 1$

$$\frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log \frac{1}{N_n(\eta)} \geqslant 0$$

while if $\sum_{k \geqslant 1} G(k) > 1$,

$$\begin{aligned} \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log \frac{1}{N_n(\eta)} &\geqslant \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log \left(\frac{1}{\sum_{k \geqslant 1} G(k)} \right)^n \\ &= \log \left(\frac{1}{\sum_{k \geqslant 1} G(k)} \right) \end{aligned}$$

Also,

$$\frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log M_n(\eta) \to \sup_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log M_n(\eta)$$

by subadditivity (see Ruelle,[11] p. 180). So

$$\begin{aligned} \lim_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log M_n(\eta) &\geqslant \sum_{\eta \in X_{0,1}} M_1(\eta) \log M_1(\eta) \\ &\geqslant \sum_{\eta \in X_{0,1}} [M_1(\eta) - 1] = 1 - 4 \end{aligned}$$

Thus

$$A(\mu) = \lim_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log M_n(\eta) + \lim_n \frac{1}{n} \sum_{\eta \in X_{0,n}} M_n(\eta) \log \frac{1}{N_n(\eta)}$$

is a well-defined expression, and is bounded below independent of μ. ∎

In the proposition below we use the fact that the derivative of $H_n(\mu_t)$ can be written as a sum of terms $C'_n(x, \eta)$ over $(x, \eta) \in B_n$ minus a sum of terms $\frac{1}{2} D'_n(x)$ over $x \in Z_{0,n}$.

Proposition 4.8. Let μ be a translation-invariant measure on $\tilde{X}$ which is symmetric in zeros and ones, and such that there exists a $\delta > 0$ so that

$$\mu\{\eta : \eta(0) = 0 \mid \eta(x) = 0 \text{ on } Z_{1,n-1} \text{ and } \eta(n) = 1\} > \delta$$

for all $n \geqslant 1$. Then for any $0 \leqslant t_1 < t_2$

$$A(\mu_{t_2}) - A(\mu_{t_1}) \leqslant \int_{t_1}^{t_2} H(\mu_t)\, dt$$

where $H \leqslant 0$ is an upper semicontinuous function.

Proof. We note that by Lemma 3.1 we have

$$\frac{dH_n(\mu_t)}{dt} = \sum_{(x,\eta) \in B_n} C'_n(x, \eta) - \frac{1}{2} \sum_{x=0}^{n} D'_n(x)$$

for all $t \geqslant 0$. Also, by Corollary 3.14

$$\limsup_n \sup_{t \in [t_1, t_2]} \frac{1}{n} \sum_{(x,\eta) \in B_n} C'_n(x, \eta) \leqslant 0$$

So

$$\begin{aligned}
A(\mu_{t_2}) - A(\mu_{t_1}) &= \lim_{n \to \infty} \frac{H_n(\mu_{t_2})}{n} - \lim_{n \to \infty} \frac{H_n(\mu_{t_1})}{n} \\
&= \lim_{n \to \infty} \frac{H_n(\mu_{t_2}) - H_n(\mu_{t_1})}{n} \\
&= \lim_{n \to \infty} \int_{t_1}^{t_2} \frac{1}{n} \frac{dH_n(\mu_t)}{dt}\, dt \\
&= \lim_{n \to \infty} \left[\int_{t_1}^{t_2} \frac{1}{n} \sum_{(x,\eta) \in B_n} C'_n(x, \eta)\, dt - \int_{t_1}^{t_2} \frac{1}{2n} \sum_{x=0}^{n} D'_n(x)\, dt \right]
\end{aligned}$$

$$\leqslant \limsup_n \int_{t_1}^{t_2} \frac{1}{n} \sum_{(x,\eta) \in B_n} C_n^t(x,\eta)\,dt - \liminf_n \int_{t_1}^{t_2} \frac{1}{2n} \sum_{x=0}^{n} D_n^t(x)\,dt$$
$$\leqslant -\liminf_n \int_{t_1}^{t_2} \frac{1}{2n} \sum_{x=0}^{n} D_n^t(x)\,dt$$
$$\leqslant -\int_{t_1}^{t_2} \liminf_n \frac{1}{2n} \sum_{x=0}^{n} D_n^t(x)\,dt$$
$$= \int_{t_1}^{t_2} \limsup_n \left(-\frac{1}{2n} \sum_{x=0}^{n} D_n^t(x)\right) dt$$

Note that $D_n(x) \geqslant 0$ for $n \geqslant 0$ and $x \in Z_{0,n}$. As observed in Liggett,[8]

$$D_m(x) \leqslant D_n(x)$$

for $m \leqslant n$ and $x \in Z_{0,m}$. Now, for $n \geqslant 0$, set

$$S(n+1) = \sum_{x=0}^{n} D_n(x)$$

Then for m, $n \geqslant 0$, by using translation invariance, we get

$$\begin{aligned} S(m+1) + S(n+1) &= \sum_{x=0}^{m} D_m(x) + \sum_{x=0}^{n} D_n(x) \\ &\leqslant \sum_{x=0}^{m} D_{m+n+2}(x) + \sum_{x=m+1}^{m+n+2} D_{m+n+2}(x) \\ &= \sum_{x=0}^{m+n+2} D_{m+n+2}(x) = S(m+n+2) \end{aligned}$$

So for $n \geqslant 1$ the sequence $S(n)$ is superadditive. Thus

$$\limsup_n \frac{-1}{2n} \sum_{x=0}^{n} D_n(x) = \lim_n \frac{-1}{2n} \sum_{x=0}^{n} D_n(x) = \inf_n \frac{-1}{n} \sum_{x=0}^{n} D_n(x)$$

which is the infimum of a sequence of upper semicontinuous functions. Hence

$$H = \limsup_n \frac{-1}{2n} \sum_{x=0}^{n} D_n(x)$$

is upper semicontinuous. ∎

Let $\mathfrak{R}$ denote the set of all reversible measures (where the definition has been extended to include δ_0 and δ_1). It follows as in Section 3 that for any translation-invariant measure μ, if $\mu \notin \mathfrak{R}$, then $H(\mu) < 0$. The next proposition can be proved using Proposition 4.8 in basically the same way that Theorem 3.15 is proven in ref. 7.

Proposition 4.9. Let ν be a translation-invariant measure such that $\nu(\tilde{X}) = 1$ if the rates are not attractive, and suppose that $\nu \notin \mathfrak{R}$. Then there exists a weakly open set G_ν containing ν and ε, $\delta > 0$ such that if μ satisfies the conditions of Proposition 4.8 and $\mu_t \in G_\nu$, then

$$A(\mu_{t+s}) - A(\mu_t) \leqslant -\delta s$$

for all $0 \leqslant s \leqslant \varepsilon$.

Proof of Theorem 4.1. Let μ be a translation-invariant measure on $\tilde{X}$ which is symmetric in zeros and ones and satisfies (4.2). Suppose that $t_n \to \infty$ and $\mu_{t_n} \to \nu$, where $\nu(\tilde{X}) = 1$ if the rates are not attractive. Then $A(\mu_{t_n})$ is a convergent sequence by Propositions 4.4 and 4.8. Thus ν must be reversible by Proposition 4.9. The case of a measure μ which is not symmetric in zeros and ones can be treated by considering the measure $\tilde{\mu}$ which we get by applying μ to configurations with zeros and ones interchanged. The ergodicity of the symmetric renewal measure is used much the same as in the proof of Theorem 1.4. ∎

From now on we will assume that the rates are attractive [$\beta(l, r)$ is monotone decreasing in l and r] in addition to (1.2) to obtain some applications of Theorem 4.1. An immediate consequence of the theorem is the following:

Corollary 4.10. Assume that $\beta(l, r)$ is monotone decreasing in l and r and satisfies (1.2). Suppose that no symmetric renewal measure exists for the rates β. Then if μ is any translation-invariant probability measure on $\tilde{X}$ which is symmetric in zeros and ones and satisfies (4.2),

$$\mu S(t) \to \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_1$$

At this point we consider rates of a form for which a symmetric renewal measure exists. Let F be as in (1.2) and suppose that there exists a $\theta > 0$ such that

$$1 < \sum_{k=1}^{\infty} \left[\frac{F(2)}{\beta(1, 1)}\right]^{1/2} \frac{\theta^k}{F(k+1)} < \infty \tag{4.11}$$

Then by attractiveness and the ratio test

$$\frac{F(k+1)}{\theta F(k)} \to \gamma \geqslant 1$$

Hence, we can choose $\theta_0 < \theta$ so that

$$1 < \sum_{k=1}^{\infty} \left[\frac{F(2)}{\beta(1, 1)}\right]^{1/2} \frac{\theta_0^k}{F(k+1)} < \infty$$

and

$$\frac{F(k+1)}{\theta_0 F(k)} \to \gamma_0 > 1$$

As shown in Chapter 6 of ref. 5, we can then construct a transient finite symmetric nearest-particle system (start the process with a finite number of ones) by setting $\beta(k, \infty) = \theta_0^k / F(k)$. If, in addition, we assume that $\sup_n \sum_{l+r=n} \beta(l, r) < \infty$, as in Section 2 of Mountford,[10] we can show that if μ is a translation-invariant measure on $\tilde{X}$ and $\mu_{t_n} \to \nu$ as $t_n \to \infty$, then ν must concentrate on $\tilde{X}$. Thus, using Theorem 4.1, we get the following result.

Corollary 4.12. Assume that $\beta(l, r)$ is monotone decreasing in l and r and satisfies (1.2). Suppose that $\sup_n \sum_{l+r=n} \beta(l, r) < \infty$ and there exists a $\theta > 0$ so that (4.11) holds for F as in (1.2). Then if μ is any translation-invariant measure on $\tilde{X}$ which satisfies (4.2) and (4.3), we have

$$\mu S(t) \to \mu_\beta$$

where μ_β is the symmetric renewal measure determined by the rates $\beta(l, r)$.

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